SOLUTIONS

VCAA 2013 Exam 1

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (5 marks)

a. If $y = x^2\log_e(x)$, find $\frac{dy}{dx}$. 2 marks

$u = x^2 \quad v = \log_e x$

$u' = 2x \quad v' = \frac{1}{x}$

$y = x^2 \log_e x$

$\frac{dy}{dx} = 2x\log_e x + x^2 \times \frac{1}{x}$

$= 2x\log_e x + x$

b. Let $f(x) = e^{x^2}$.

Find $f'(3)$. 3 marks

$y = e^{x^2}$

$\frac{dy}{dx} = 2x\, e^{x^2}$

$\therefore f'(3) = 2 \times 3 e^9$

$= 6e^9$

[Longer method: let $u = x^2$ $\quad \frac{du}{dx} = 2x$

$y = e^u$ $\quad \frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\therefore \frac{dy}{dx} = e^u \cdot 2x = 2xe^{x^2}$

TURN OVER

Question 4 (2 marks)

Solve the equation $\sin\left(\frac{x}{2}\right) = -\frac{1}{2}$ for $x \in [2\pi, 4\pi]$.

$$\sin\left(\frac{x}{2}\right) = -\frac{1}{2}, \qquad 2\pi \le x \le 4\pi$$
$$\therefore \frac{x}{2} = \frac{7\pi}{6}, \frac{11\pi}{6} \qquad \pi \le \frac{x}{2} \le 2\pi$$
$$x = \frac{7\pi}{3}, \frac{11\pi}{3}$$

Question 5 (4 marks)

a. Solve the equation $2\log_3(5) - \log_3(2) + \log_3(x) = 2$ for x. 2 marks

$$\log_3 25 - \log_3 2 + \log_3 x = 2$$
$$\therefore \log_3\left(\frac{25x}{2}\right) = 2$$
$$\frac{25x}{2} = 3^2$$
$$\therefore x = \frac{18}{25}$$

b. Solve the equation $3^{-4x} = 9^{6-x}$ for x. 2 marks

$$3^{-4x} = 9^{6-x}$$
$$\therefore 3^{-4x} = (3^2)^{6-x}$$
$$3^{-4x} = 3^{12-2x}$$
$$\therefore -4x = 12-2x$$
$$-12 = 2x$$
$$x = -6$$

Question 2 (2 marks)

Find an anti-derivative of $(4 - 2x)^{-5}$ with respect to x.

$$\int (4-2x)^{-5}\, dx$$

$$= \frac{(4-2x)^{-4}}{-4 \times -2}$$

$$= \frac{(4-2x)^{-4}}{8} = \frac{1}{8(4-2x)^4}$$

Question 3 (2 marks)

The function with rule $g(x)$ has derivative $g'(x) = \sin(2\pi x)$.

Given that $g(1) = \frac{1}{\pi}$, find $g(x)$.

$$g'(x) = \sin(2\pi x)$$

$$\therefore g(x) = \int \sin(2\pi x)\, dx$$

$$= -\frac{1}{2\pi} \cos(2\pi x) + c$$

When $x = 1$, $g(1) = \frac{1}{\pi}$

$$\therefore \frac{1}{\pi} = -\frac{1}{2\pi} \cos(2\pi) + c$$

$$\therefore \frac{1}{\pi} = \frac{-1}{2\pi} + c$$

$$\therefore c = \frac{3}{2\pi}$$

$$\therefore g(x) = -\frac{1}{2\pi} \cos(2\pi x) + \frac{3}{2\pi}$$

Question 6 (3 marks)

Let $g: R \to R$, $g(x) = (a - x)^2$, where a is a real constant.

The average value of g on the interval $[-1, 1]$ is $\frac{31}{12}$.

Find all possible values of a.

$$\frac{1}{1-{-1}} \int_{-1}^{1} (a-x)^2 \, dx = \frac{31}{12}$$

$$\therefore \frac{1}{2}\left[\frac{(a-x)^3}{-3}\right]_{-1}^{1} = \frac{31}{12}$$

$$\therefore \frac{(a-1)^3}{-3} + \frac{(a+1)^3}{3} = \frac{31}{6}$$

$$(a+1)^3 - (a-1)^3 = \frac{31}{2}$$

$$(a^3+3a^2+3a+1) - (a^3-3a^2+3a-1) = \frac{31}{2}$$

$$6a^2 + 2 = \frac{31}{2}$$

$$\therefore 6a^2 = \frac{27}{2}$$

$$a^2 = \frac{27}{12}$$

$$\therefore a^2 = \frac{9}{4}$$

$$a = \pm\sqrt{\frac{9}{4}}$$

$$\therefore a = \pm\frac{3}{2}$$

Question 7 (6 marks)

The probability distribution of a discrete random variable, X, is given by the table below.

x	0	1	2	3	4
$\Pr(X = x)$	0.2	$0.6p^2$	0.1	$1 - p$	0.1

a. Show that $p = \frac{2}{3}$ or $p = 1$. 3 marks

$$0.2 + 0.6p^2 + 0.1 + 1 - p + 0.1 = 1$$

$$0.6p^2 - p + 0.4 = 0$$

$$6p^2 - 10p + 4 = 0$$

$$3p^2 - 5p + 2 = 0$$

$$(3p-2)(p-1) = 0$$

$$p = \frac{2}{3}, 1$$

b. Let $p = \frac{2}{3}$.

i. Calculate $E(X)$. 2 marks

x	0	1	2	3	4
$\Pr(X=x)$	0.2	$\frac{4}{15}$	0.1	$\frac{1}{3}$	0.1

$0.6 \times \left(\frac{2}{3}\right)^2$

$= \frac{3}{5} \times \frac{4}{9}$

$= \frac{4}{15}$

$E(X) = 0 \times 0.2 + 1 \times \frac{4}{15} + \frac{2}{10} + 1 + \frac{4}{10}$

$= \frac{4}{15} + \frac{1}{5} + 1 + \frac{2}{5}$

$= 1 + \frac{4}{15} + \frac{3}{15} + \frac{6}{15} = \frac{28}{15}$

ii. Find $\Pr(X \geq E(X))$. 1 mark

$\Pr\left(X \geq \frac{28}{15}\right)$

$= \Pr(X=2) + \Pr(X=3) + \Pr(X=4)$

$= 0.1 + \frac{1}{3} + 0.1$

$= \frac{1}{5} + \frac{1}{3}$

$= \frac{8}{15}$

Question 8 (3 marks)

A continuous random variable, X, has a probability density function

$$f(x)=\begin{cases}\frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right) & \text{if } x\in[0,\ 2]\\ 0 & \text{otherwise}\end{cases}$$

Given that $\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right)=\frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)+\sin\left(\frac{\pi x}{4}\right)$, find $E(X)$.

$$\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right)=\frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)+\sin\left(\frac{\pi x}{4}\right)$$

Now, $E(X)=\int_0^2 x\frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right)dx$

$$x\sin\left(\frac{\pi x}{4}\right)=\int\frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx+\int\sin\left(\frac{\pi x}{4}\right)dx$$

$$\therefore x\sin\left(\frac{\pi x}{4}\right)=\int\frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx-\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)$$

$$\therefore x\sin\left(\frac{\pi x}{4}\right)+\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)=\int\frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx$$

$$\therefore\left[x\sin\left(\frac{\pi x}{4}\right)+\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)\right]_0^2=\int_0^2\frac{\pi x}{4}\cos\left(\frac{\pi x}{4}\right)dx$$

$$\therefore E(X)=\left[x\sin\left(\frac{\pi x}{4}\right)+\frac{4}{\pi}\cos\left(\frac{\pi x}{4}\right)\right]_0^2$$

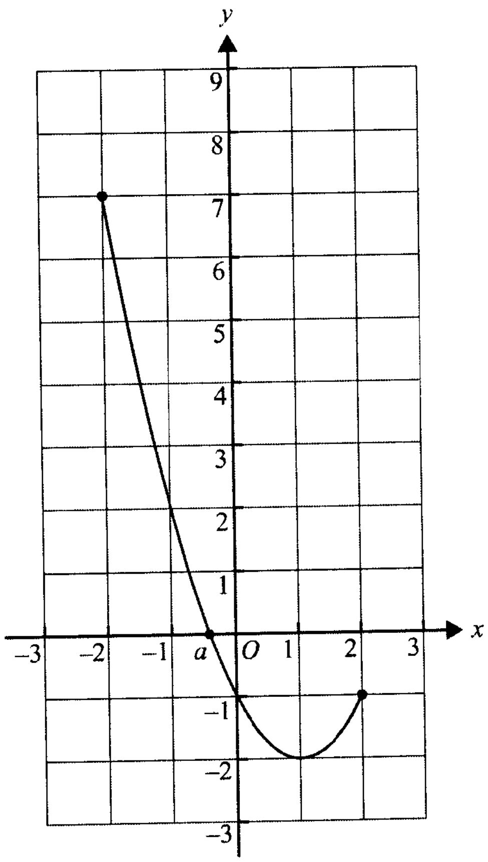
$$=\left[2\sin\left(\frac{\pi}{2}\right)+\frac{4}{\pi}\cos\left(\frac{\pi}{2}\right)\right]$$

$$-\left[0.\sin 0+\frac{4}{\pi}\cos(0)\right]$$

$$=2-\frac{4}{\pi}$$

Question 9 (6 marks)

The graph of $f(x) = (x - 1)^2 - 2$, $x \in [-2, 2]$, is shown below. The graph intersects the x-axis where $x = a$.



a. Find the value of a. 1 mark

$(x-1)^2 - 2 = 0$

$(x-1)^2 = 2$

$x - 1 = \pm\sqrt{2}$

$\therefore x = 1 \pm \sqrt{2}$

But $a < 1$ $\therefore$ $a = 1 - \sqrt{2}$

c. The following sequence of transformations is applied to the graph of the function $f(x)$

- a translation of one unit in the negative direction of the x-axis
- a translation of one unit in the negative direction of the y-axis
- a dilation from the x-axis of factor $\frac{1}{3}$

Find

i. the rule of the image of ~~g~~ f after the sequence of transformations has been applied **2 marks**

$f(x) = (x-1)^2 - 2$

$f_1(x) = f(x+1) = (x+1-1)^2 - 2 = x^2 - 2$

$f_2(x) = f_1(x) - 1 = x^2 - 2 - 1 = x^2 - 3$

$f_3(x) = \frac{1}{3} f_2(x) = \frac{1}{3}(x^2 - 3)$

$= \frac{x^2}{3} - 1$

$\therefore$ Image: $y = \frac{x^2}{3} - 1$

ii. the domain of the image of f after the sequence of transformations has been applied. **1 mark**

Original domain was $[-2, 2]$

$(-2, 0) \rightarrow (-3, 0) \rightarrow (-1, -1) \rightarrow (-3, -\frac{1}{3})$

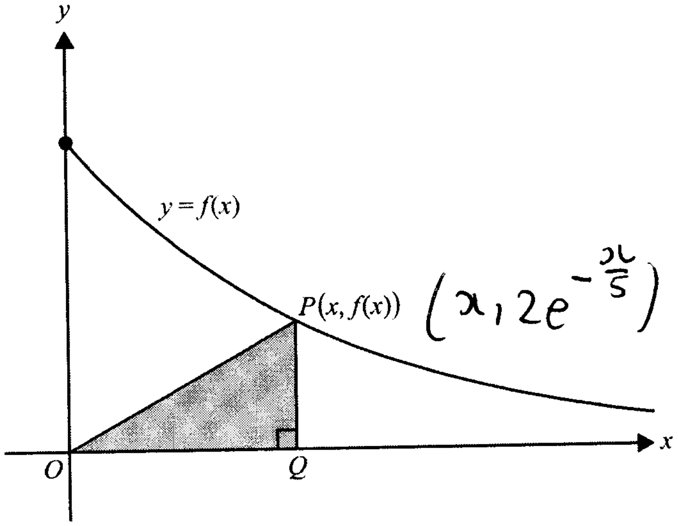
$(2, 0) \rightarrow (1, 0) \rightarrow (1, -1) \rightarrow (1, -\frac{1}{3})$

New domain: $[-3, 1]$

Question 10 (7 marks)

Let $f: [0, \infty) \to R$, $f(x) = 2e^{-\frac{x}{5}}$.

A right-angled triangle OQP has vertex O at the origin, vertex Q on the x-axis and vertex P on the graph of f, as shown. The coordinates of P are $(x, f(x))$.



a. Find the area, A, of the triangle OQP in terms of x. 1 mark

$$A = \frac{x\,f(x)}{2}$$

$$= \frac{x \cdot 2e^{-\frac{x}{5}}}{2}$$

$$= xe^{-x/5}$$

b. Find the maximum area of triangle OQP and the value of x for which the maximum occurs. 3 marks

$A(x) = xe^{-x/5}$ $\quad u = x \quad v = e^{-x/5}$

$u' = 1 \quad v' = -\frac{1}{5}e^{-x/5}$

$A'(x) = \frac{-x}{5}e^{-x/5} + e^{-x/5}$

For a maximum, $A'(x) = 0$

$\therefore e^{-x/5} - \frac{x}{5}e^{-x/5} = 0$

$e^{-x/5}\left(1 - \frac{x}{5}\right) = 0$

$\therefore x = 5$

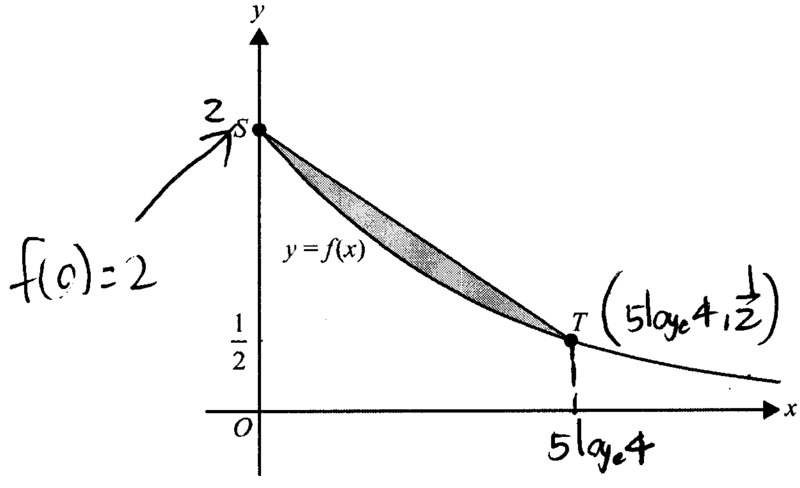
$(e^{-x/5} \neq 0$ for all $x \in R)$

When $x = 5$, $A(5) = 5e^{-5/5} = 5e^{-1}$

$\therefore$ Maximum area $= \frac{5}{e}$ sq. units when $x = 5$.

c. Let S be the point on the graph of f on the y-axis and let T be the point on the graph of f with the y-coordinate $\frac{1}{2}$.

Find the area of the region bounded by the graph of f and the line segment ST. 3 marks



$2e^{-x/5} = \frac{1}{2}$

$\therefore e^{-x/5} = \frac{1}{4}$

$-\frac{x}{5} = \log_e \frac{1}{4}$

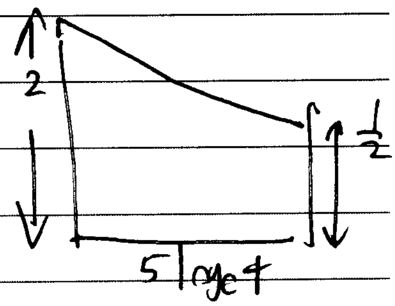
$x = 5\log_e 4$

$A_{trapezium}$

$= \frac{5\log_e 4}{2}\left(2 + \frac{1}{2}\right)$

$= \frac{5}{2}\log_e 4 \times \frac{5}{2}$

$= \frac{25}{4}\log_e 4$



Shaded area $= \frac{25}{4}\log_e 4 - \int_0^{5\log_e 4} 2e^{-x/5}\,dx$

$= \frac{25}{4}\log_e 4 + \left[10e^{-x/5}\right]_0^{5\log_e 4}$

$= \frac{25}{4}\log_e 4 + \left(10e^{-\log_e 4} - 10\right)$

$= \frac{25}{4}\log_e 4 + 10 \times \frac{1}{4} - 10$

$= \frac{25}{4}\log_e 4 - \frac{30}{4}$

$= \frac{25}{4}\log_e 4 - \frac{15}{2}$