Y12 Mid-Year

MATHEMATICAL METHODS (CAS) UNITS 3 & 4

June 2014

Reading time: 10 minutes
Writing time: 90 minutes
Total time: 2 periods

Structure of the book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*	*Time (minutes)*
A: Short answer Questions	**5**	**25**	**25**	**35**
B: : Multiple–Choice Questions	**15**	**15**	**15**	**25**
C: Analysis Questions	**3**	**3**	**35**	**30**
			Total =75	**90 minutes**

Instructions to students

This exam consists of two papers: one for Section A (green) and another paper for Section B and Section C (white).
All questions in all sections should be answered.
Diagrams in this exam are not to scale except where otherwise stated.
Where more than one mark is allocated to a question, appropriate working must be shown.
If a question requires a numerical answer then an exact value must be given unless a decimal approximation is specifically asked for.

- Students are permitted to bring the following items into the examination: pens, pencils, highlighters, erasers, sharpeners, rulers, one bound reference book that may be annotated (can be typed, handwritten or a textbook), one approved graphics calculator (memory DOES NOT have to be cleared) and, if desired, one scientific calculator.
- Students are NOT permitted to bring blank sheets of paper or white out liquid/tape into the examination.

Materials provided

- A question book of 22 pages, with an answer sheet for the multiple-choice questions.
- A separate sheet with miscellaneous formulas.
- Working space is provided throughout the question book.

Instructions

- Write your **name** in the box provided on the multiple-choice answer sheet.
- Calculators are allowed for Section B & C only.

Students are not permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination.

Section A: Short answer Questions (CAS FREE) (25 marks)

Where an exact answer is required a decimal approximation will not be accepted.
Where more than one mark is allocated to a question, appropriate working must be shown.
Diagrams in this trial exam are not drawn to scale.
A formula sheet is provided.
Answer all questions

Question 1

a. Let $f(x) = x\log_e(x^2+5)$. Find $f'(x)$.

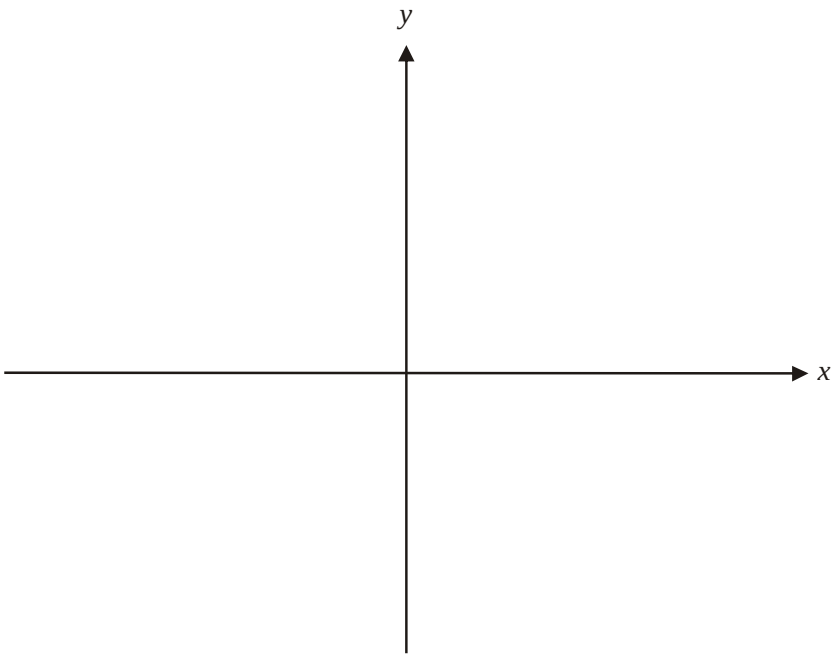
b. Let $y = \dfrac{\tan(x)}{e^{2x}}$. Evaluate $\dfrac{dy}{dx}$ when $x = 0$.

2+3 = 5 marks

Question 2

Let $f: R \to R,\ f(x) = \left|x^2 - 6x + 5\right|$.

a. Sketch the graph of $y = f(x)$ on the set of axes below. Indicate clearly any axes intercepts or turning points.



b. Write down the domain of the derivative function f'.

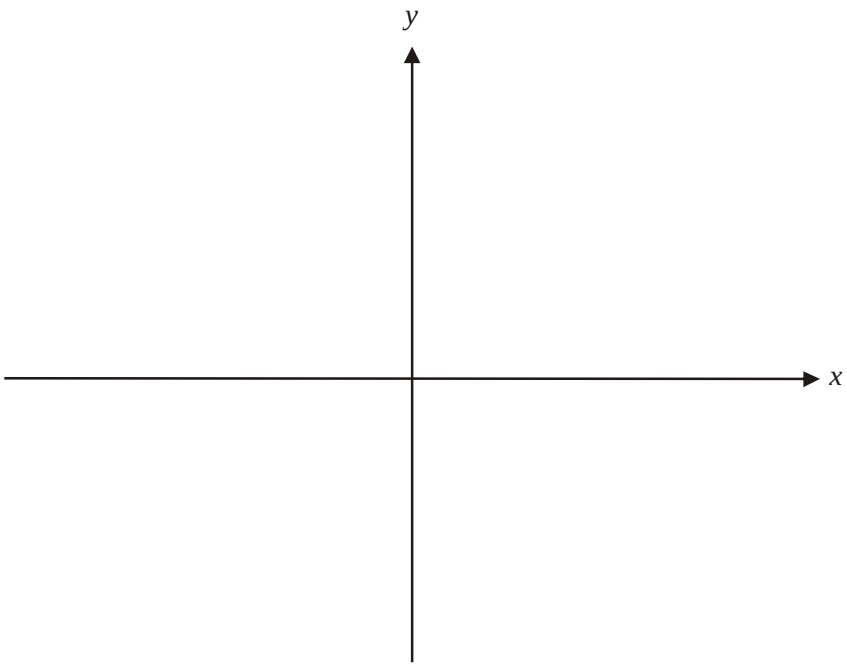
c. Write down the values of x for which $f'(x) > 0$.

3 + 1 + 1 = 5 marks

Question 3

Let $h:(2,\infty)\to R, h(x)=\dfrac{1}{x-2}+1$.

a. On the axes below, sketch the graph of $y=h(x)$. Label any asymptotes with their equation.

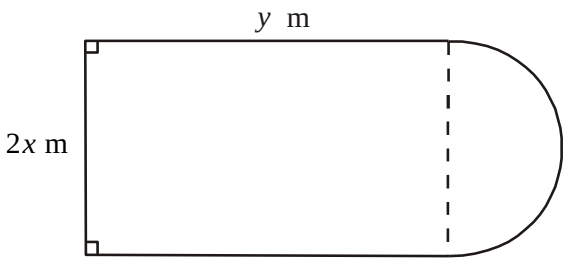


b. Find the rule and the domain of the inverse function h^{-1}.

2 + 3 = 5 marks

Question 4

A pool complex is made up of a rectangular swimming pool with side lengths $2x$ m and y m attached to a semi-circular spa of radius x m.



The perimeter of the pool complex is 100m.

a. Express y in terms of x.

b. Show that the surface area of the pool complex is given by

$$A = 100x - \frac{x^2}{2}(\pi + 4).$$

c. Find the value of x for which the surface area of the pool complex is a maximum. It is not necessary to find this maximum surface area.

2+1+2 = 5 marks

Question 5

Let $h: R \to R,\ h(x) = \sqrt{2}\sin(2x)$.

a. Find the range of *h*.

(1 mark)

b. Find the General Solution of the equation $h(x) = -1$ for $x \in R$.

(4 marks)

End of Section A: Short answer Questions – CAS Free

SECTION B (15 marks)

Instruction for Section A

Answer all questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

SECTION 1

Question 1

$g:[-1,2] \to R,\ g(x) = |x-1| - 2$.
The range of g is

A. $[-2, 0]$
B. $[0, 2]$
C. $[-2, 1]$
D. $[-1, 2]$
E. $[-2, \infty)$

Question 2

The maximal domain and the range of the function $f(x) = \frac{5}{x^2} - k$, where k is a positive real number, are given by

A. $d_f = R \setminus \{0\}$ and $r_f = (-k, \infty)$
B. $d_f = R \setminus \{5\}$ and $r_f = R \setminus \{-k\}$
C. $d_f = R \setminus \{0\}$ and $r_f = (k, \infty)$
D. $d_f = R \setminus \{5\}$ and $r_f = R \setminus \{k\}$
E. $d_f = R \setminus \{0\}$ and $r_f = (-\infty, -k)$

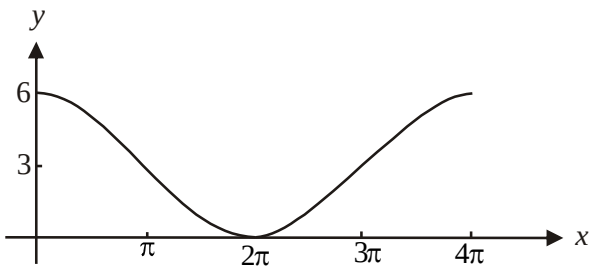
Question 3

The solutions to the equation $4\sin(2x) + 1 = 3.6$ for $x \in [0, 360^\circ]$ are closest to

A. 20.3° and 69.7° only
B. 40.5° and 139.5° only
C. 0.35°, 1.22°, 3.50° and 4.36°
D. 20.3°, 69.7°, 200.3° and 249.7°
E. 40.5°, 139.5°, 400.5° and 499.5°

Question 4

The graph showing one period of a function is shown below.



The rule for this function could be

A. $y = -3\cos\left(\frac{x}{2}\right) + 3$

B. $y = -3\cos(2x) + 6$

C. $y = 3\cos\left(\frac{x}{2}\right) + 3$

D. $y = 3\cos(2x) + 3$

E. $y = 6\cos(2x) + 6$

Question 5

$\left(\frac{x^3 y^{-2}}{x^{-2} y^5}\right)^{-2} \times \sqrt[3]{\frac{y^6}{x^{12}}}$ can be simplified to

A. $\frac{1}{x^6 y}$

B. $\frac{1}{x^6 y^4}$

C. $\frac{1}{x^{14} y^4}$

D. $\frac{y^{16}}{x^{14}}$

E. $\frac{y^{12}}{x^{38}}$

Question 6

The average rate of change of the function $g(x) = \left|\cos\left(2x - \frac{\pi}{3}\right)\right|$ between $x = 0$ and $x = \frac{\pi}{4}$ is

A. $\frac{\sqrt{3}}{2}$

B. $\frac{\sqrt{3}}{2} - \frac{1}{8\pi}$

C. $\frac{\sqrt{3}}{2} - \frac{1}{2}$

D. $\frac{\sqrt{3} - 1}{8\pi}$

E. $\frac{2\left(\sqrt{3} - 1\right)}{\pi}$

Question 7

$\lim\limits_{x \to -1} \frac{x^2 - 4x - 5}{x + 1}$ is equal to

A. -8

B. -6

C. -4

D. 0

E. 3

Question 8

The gradient of the normal to the curve $y = \frac{a}{2x-5} + 1$ at the point where $x = 3,$ is $\frac{1}{2}$.
The value of *a* is

A. $-\frac{1}{4}$

B. $-\frac{1}{2}$

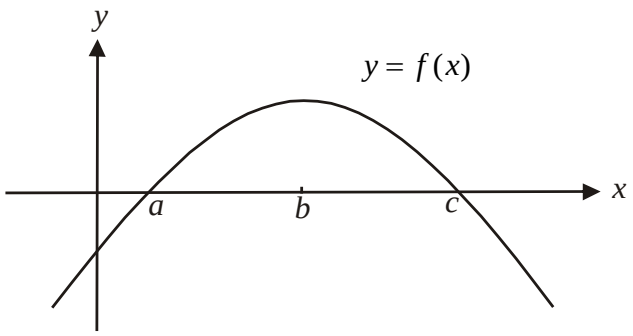
C. $\frac{1}{2}$

D. 1

E. 2

Question 9

The graph of the quadratic function $y = f(x)$ is shown below.



Which one of the following statements is true?

A. $f(b) = 0$

B. $f(a) < 0$

C. $f(0) > 0$

D. $f'(x) > 0 \text{ for all } x$

E. $f'(c) < 0$

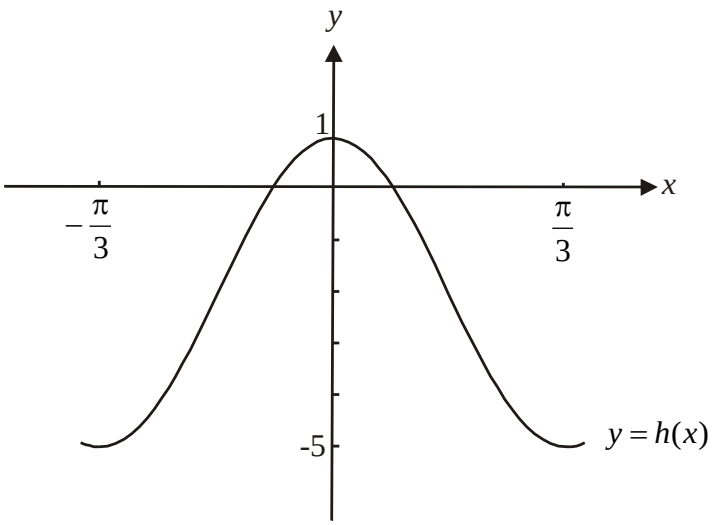
Question 10

A spherical piece of ice with volume V (in mm^3) is melting at the rate of 11.52 π mm^3 per minute. When the radius of the sphere is decreasing at the rate of 2mm per minute, the radius of the piece of ice, in mm, is

A. 0.69
B. 0.80
C. 1.20
D. 2.73
E. 5.71

Question 11

Part of the graph of the function h is shown below.



The equation of this graph could be

A. $y = 3\cos(2x) - 2$

B. $y = \sin\left(2x - \frac{\pi}{3}\right) - 4$

C. $y = 2\cos\left(3x + \frac{\pi}{4}\right) - 3$

D. $y = 3\sin 3\left(x + \frac{\pi}{6}\right) - 2$

E. $y = 3\cos 3\left(x - \frac{\pi}{3}\right) - 2$

Question 12

For the function *f*,

$$f\left(\frac{\pi}{2}+\theta\right)=f\left(\frac{\pi}{2}-\theta\right)$$

and $f\left(\frac{\pi}{2}+\theta\right)=-f\left(\theta-\frac{\pi}{2}\right)$ for all real values of θ.

The rule for *f* could be

A. $f(x)=\tan(x)$
B. $f(x)=-\tan(x)$
C. $f(x)=\cos(x)$
D. $f(x)=-\cos(x)$
E. $f(x)=\sin(x)$

Question 13

The function *h* can be differentiated for all real values of *x*.
The derivative of the function $h(\cos(2x))$ is given by

A. $h'(\cos(2x))$
B. $h'(-2\sin(2x))$
C. $-2h'(\cos(2x))\sin(2x)$
D. $2\sin(2x)h'(\cos(2x))$
E. $2\sin(2x)\ h(\cos(2x))$

Question 14

If $y=|1-x|$, then the rate of change of *y* with respect to *x* at $x=2$ is

A. -2
B. -1
C. 0
D. 1
E. 2

Question 15

The simultaneous linear equations

$$\begin{aligned}5x+(a-3)y&=1\\ ax+2y&=a\end{aligned}$$

where $a \in R$, will have no solutions for

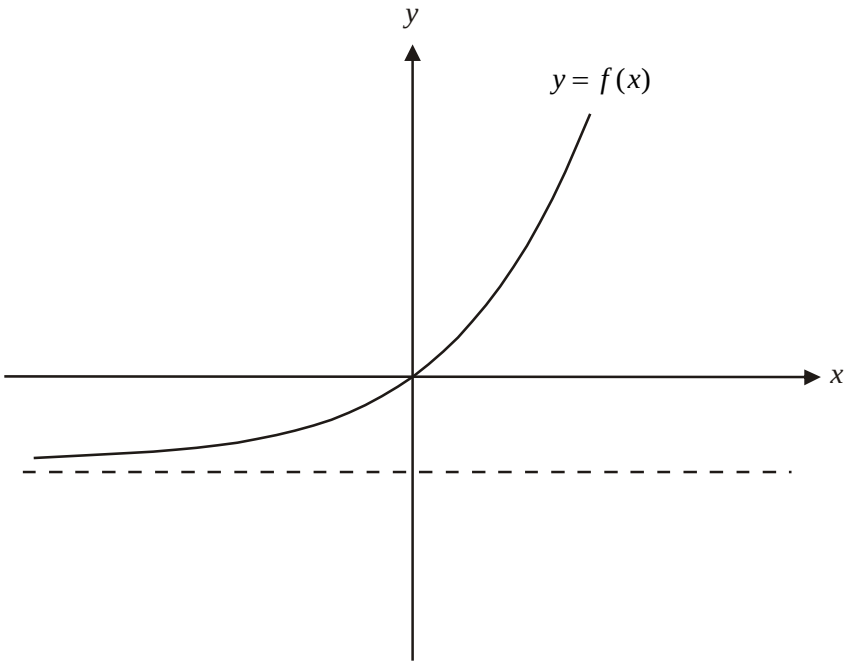
A. $a=-2$
B. $a=5$
C. $a \in R\setminus\{-2\}$
D. $a \in R\setminus\{-2,5\}$
E. $a \in \{-2,5\}$

End Of Section B

Section C: Analysis Questions (35 marks)

Question 1 (11 marks)

The graph of the function $f : R \to R, f(x) = e^{ax} - b$ has a horizontal asymptote and passes through the origin.
The graph of $y = f(x)$ is shown below.



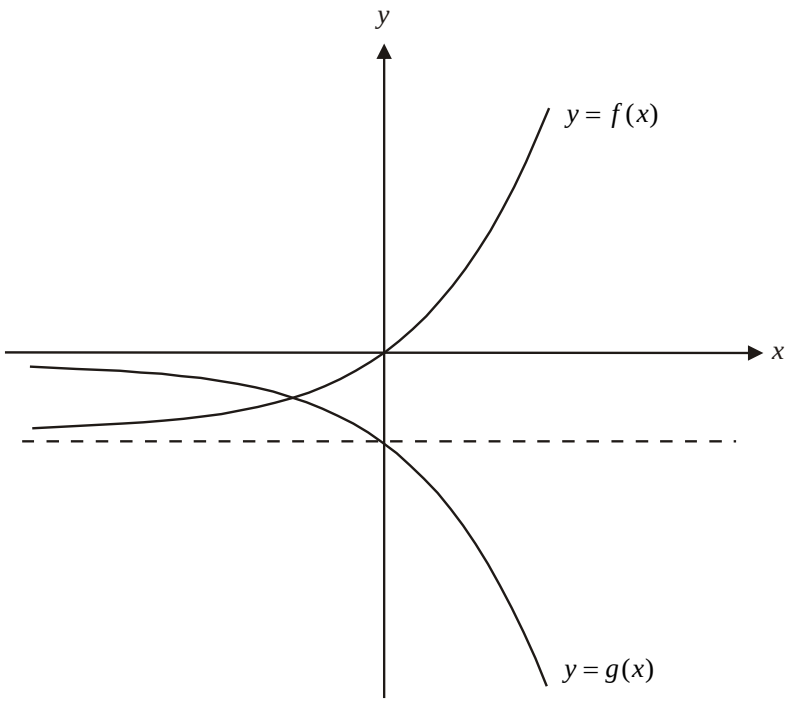
a. Show that $b = 1$.

1 mark

b. Given that $f'(1) = 2e^2$, show that $a = 2$.

2 marks

The graph of the function $y = f(x)$ undergoes two transformations. The graph of the original function $y = f(x)$, together with the graph of the final function $y = g(x)$, are shown below.

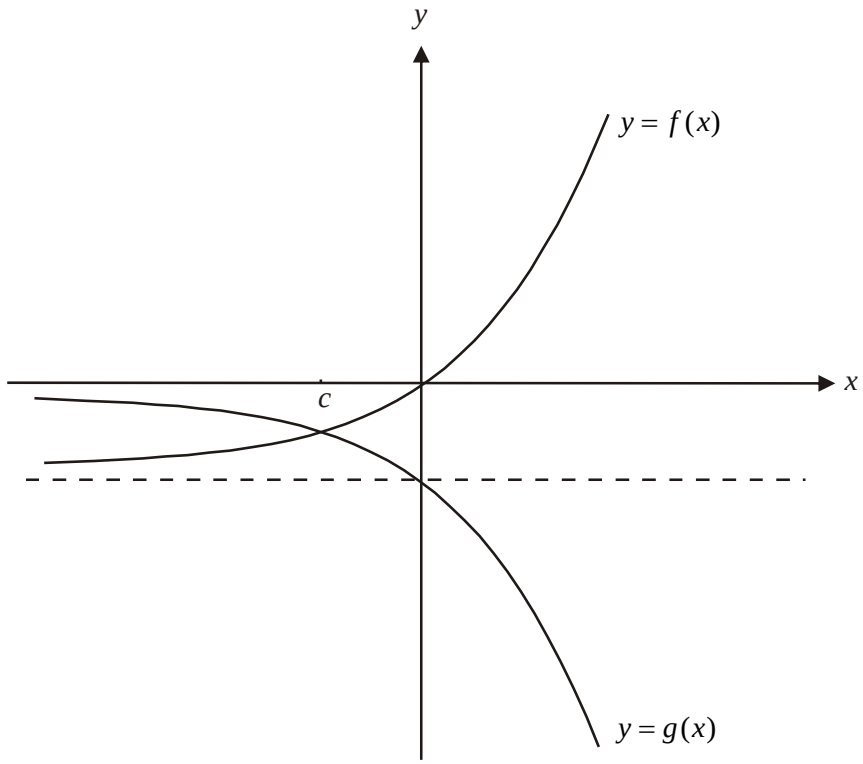


c. **i.** Describe the two transformations that the graph of $y = f(x)$ has undergone.

ii. Show that $g(x) = -e^{2x}$.

$2 + 2 = 4$ marks

The graphs of $y = f(x)$ and $y = g(x)$ intersect at the point where $x = c$ as shown below.



d. Show that $c = -\frac{1}{2}\log_e(2)$.

2 marks

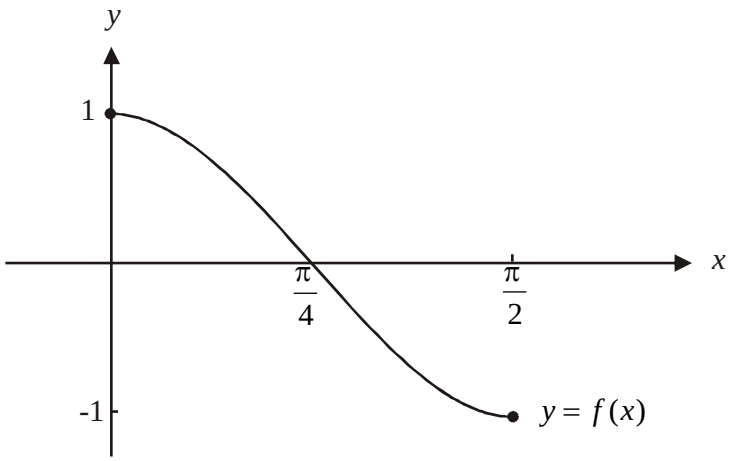
e. **Solve** $f(x) = -f(\frac{x}{2})$

2 marks

Total 11 marks

Question 2 (12 marks)

The graph of the function $f:\left[0,\frac{\pi}{2}\right]\rightarrow R, f(x)=\cos(2x)$ is shown below.



a. Find $f'(x)$.

1 mark

b. Find the coordinates of the point(s) where the gradient of the tangent to the graph of $y=f(x)$ is – 1. Express the coordinates as exact values.

2 marks

A tangent to the graph of $y = f(x)$ has an x-intercept of $\frac{\pi + \sqrt{3}}{6}$ and a y-intercept of $\frac{\sqrt{3}\pi + 3}{6}$.

c. **i.** Show that the gradient of this tangent is $-\sqrt{3}$.

_____

_____

_____

_____

_____

ii. Using your answer to part **a.** find the values of x where the function f has a gradient of $-\sqrt{3}$.

_____

_____

_____

_____

iii. Hence find the coordinates of the point of tangency.

_____

_____

_____

_____

_____

$1 + 2 + 2 = 5$ marks

d. Find the maximum and minimum values of $|f(x)|$.

2 marks

e. Let $g: R \to R, g(x) = \cos(2x)$.
Find the general solution for x of the equation $g(x) = 0 \cdot 5$.

2 marks

Total 12 marks

Question 3 (12 marks)

Let $f: R^+ \cup \{0\} \to R,\ f(x) = -x^2 + 3x + 2.$

a. Over what interval is the graph of *f* strictly increasing?

(2 marks)

The graph of *f* undergoes a transformation, $T: R^2 \to R^2$, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$$

and where *c* and *d* are real numbers. The graph of the image function has its turning point located at the origin $O(0,0)$.

b. Find the values of *c* and *d*.

(2 marks)

Suppose the graph of *f* is dilated by a factor of *k* from one of the axes. The image graph passes through the point $\left(2\left(3+\sqrt{17}\right),\ 0\right)$.

c. Find the value of *k*.

(2 marks)

d. Let $h: R \to R,\ h(x) = |x|$ and let $g(x) = f(h(x))$.

i. Write down the rule for g.

(1 mark)

ii. Explain why g exists.

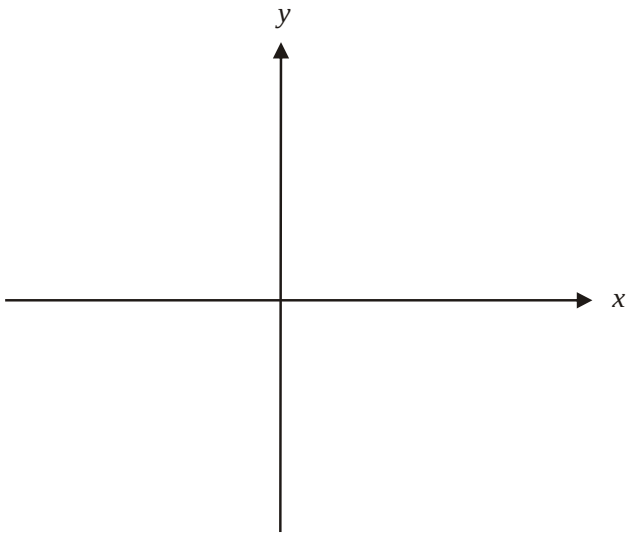
(1 mark)

iii. Find $g'(x)$.

(2marks)

iv. Sketch the graph of $y = g'(x)$ on the set of axes below.

(2 marks)



Total = 12 marks

End of Section C

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Mathematical Methods (CAS) Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

MATHEMATICAL METHODS (cas)

MID-YEAR EXAMINATION – JUNE 2014

MULTIPLE-CHOICE ANSWER SHEET

STUDENT NAME:----------

INSTRUCTIONS

Fill in the letter that corresponds to your choice. Example: (A) ■ (C) (D) (E)

The answer selected is B. Only one answer should be selected.

1. (A) (B) (C) (D) (E)
2. (A) (B) (C) (D) (E)
3. (A) (B) (C) (D) (E)
4. (A) (B) (C) (D) (E)
5. (A) (B) (C) (D) (E)
6. (A) (B) (C) (D) (E)
7. (A) (B) (C) (D) (E)
8. (A) (B) (C) (D) (E)
9. (A) (B) (C) (D) (E)
10. (A) (B) (C) (D) (E)
11. (A) (B) (C) (D) (E)
12. (A) (B) (C) (D) (E)
13. (A) (B) (C) (D) (E)
14. (A) (B) (C) (D) (E)
15. (A) (B) (C) (D) (E)