SECTION A – Multiple-choice questions

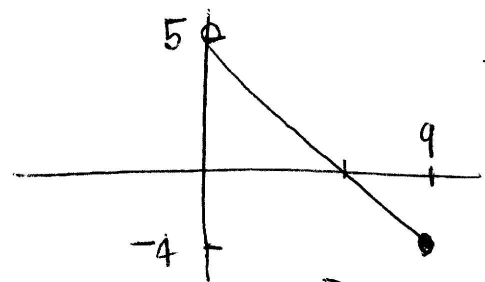
Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The linear function $f: D \to R,\ f(x) = 5 - x$ has range $[-4, 5)$.
The domain D is

A. $(0, 9]$
B. $(0, 1]$
C. $[5, -4)$
D. $[-9, 0)$
E. $[1, 9)$



$-4 = 5 - x$
$\therefore x = 9$

Domain: $(0, 9]$

Question 2

Let $f: R \to R,\ f(x) = 1 - 2\cos\left(\frac{\pi x}{2}\right)$.

The period and range of this function are respectively

A. 4 and $[-2, 2]$
B. 4 and $[-1, 3]$
C. 1 and $[-1, 3]$
D. 4π and $[-1, 3]$
E. 4π and $[-2, 2]$

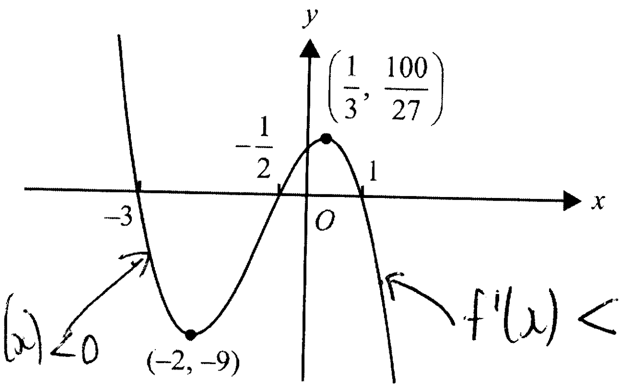
$+2 \uparrow$ 3
1
$-2 \downarrow$ −1

Range: $[-1, 3]$

Period: $\dfrac{2\pi}{\frac{\pi}{2}} = 2\pi \times \dfrac{2}{\pi}$
$= 4$

Question 3

Part of the graph $y = f(x)$ of the polynomial function f is shown below.



$f'(x) < 0$ $\quad$ $f'(x) < 0$

$(-\infty, -2) \cup \left(\frac{1}{3}, \infty\right)$

$f'(x) < 0$ for

A. $x \in (-2, 0) \cup \left(\frac{1}{3}, \infty\right)$

B. $x \in \left(-9, \frac{100}{27}\right)$

C. $x \in (-\infty, -2) \cup \left(\frac{1}{3}, \infty\right)$

D. $x \in \left(-2, \frac{1}{3}\right)$

E. $x \in (-\infty, -2] \cup (1, \infty)$

Question 4

The average rate of change of the function f with rule $f(x) = 3x^2 - 2\sqrt{x+1}$, between $x = 0$ and $x = 3$, is

A. 8

B. 25

C. $\frac{53}{9}$

D. $\frac{25}{3}$

E. $\frac{13}{9}$

$$\frac{f(3) - f(0)}{3 - 0}$$

$$f(3) = 3 \times 3^2 - 2\sqrt{4}$$
$$= 27 - 4 = 23$$

$$f(0) = -2\sqrt{1} = -2$$

$$\therefore \frac{f(3) - f(0)}{3} = \frac{23 - -2}{3} = \frac{25}{3}$$

Question 5

Which one of the following is the inverse function of $g: [3, \infty) \to R$, $g(x) = \sqrt{2x-6}$?

Eliminate A, E due to domain.

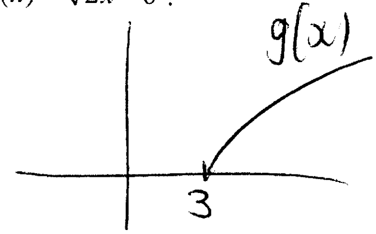
~~A.~~ $g^{-1}: [3, \infty) \to R$, $g^{-1}(x) = \dfrac{x^2+6}{2}$

B. $g^{-1}: [0, \infty) \to R$, $g^{-1}(x) = (2x-6)^2$

C. $g^{-1}: [0, \infty) \to R$, $g^{-1}(x) = \sqrt{\dfrac{x}{2}+6}$

D. $g^{-1}: [0, \infty) \to R$, $g^{-1}(x) = \dfrac{x^2+6}{2}$

~~E.~~ $g^{-1}: R \to R$, $g^{-1}(x) = \dfrac{x^2+6}{2}$



$y = \sqrt{2x-6}$

$x = \sqrt{2y-6}$

$x^2 = 2y-6$

$2y = x^2+6$

$y = \dfrac{x^2}{2}+3$

dom(g) $[3, \infty)$ ran(g) $[0, \infty)$

dom(g^{-1}) $[0, \infty)$ ran(g^{-1}) $[3, \infty)$

Question 6

Consider the graph of the function defined by $f: [0, 2\pi] \to R$, $f(x) = \sin(2x)$.

The square of the length of the line segment joining the points on the graph for which $x = \dfrac{\pi}{4}$ and $x = \dfrac{3\pi}{4}$ is

A. $\dfrac{\pi^2+16}{4}$

B. $\pi + 4$

C. 4

D. $\dfrac{3\pi^2+16\pi}{4}$

E. $\dfrac{10\pi^2}{16}$

$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$f\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$

$\therefore$ Points are: $\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right)$

$d^2 = \left(\frac{3\pi}{4} - \frac{\pi}{4}\right)^2 + (-1-1)^2$

$= \left(\frac{\pi}{2}\right)^2 + 4 = \frac{\pi^2}{4} + 4 = \frac{\pi^2+16}{4}$

Question 7

The number of pets, X, owned by each student in a large school is a random variable with the following discrete probability distribution.

x	0	1	2	3
$\Pr(X = x)$	0.5	0.25	0.2	0.05

If two students are selected at random, the probability that they own the same number of pets is

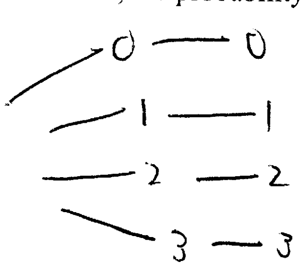
A. 0.3

B. 0.305

C. 0.355

D. 0.405

E. 0.8



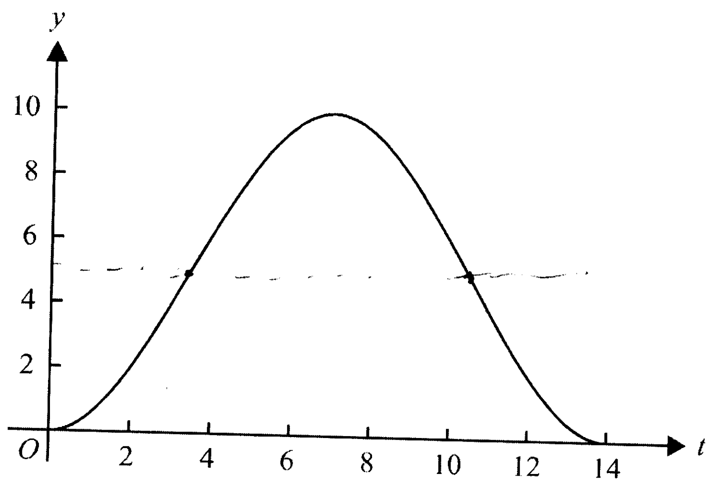
Pr(same pet)

$= (0.5)^2 + (0.25)^2 + (0.2)^2 + (0.05)^2$

$= 0.355$

Question 8

The UV index, y, for a summer day in Melbourne is illustrated in the graph below, where t is the number of hours after 6 am.



10

5

0

$a = 5$

Shape:
Negative cosine
Period $= 14$

$\therefore \frac{2\pi}{n} = 14$

$\therefore n = \frac{\pi}{7}$

The graph is most likely to be the graph of

A. $y = 5 + 5\cos\left(\frac{\pi t}{7}\right)$

B. $y = 5 - 5\cos\left(\frac{\pi t}{7}\right)$

C. $y = 5 + 5\cos\left(\frac{\pi t}{14}\right)$

D. $y = 5 - 5\cos\left(\frac{\pi t}{14}\right)$

E. $y = 5 + 5\sin\left(\frac{\pi t}{14}\right)$

$y = 5 - 5\cos\left(\frac{\pi t}{7}\right)$

Question 9

Given that $\frac{d(xe^{kx})}{dx} = (kx+1)e^{kx}$, then $\int xe^{kx}\,dx$ is equal to

A. $\frac{xe^{kx}}{kx+1} + c$

B. $\left(\frac{kx+1}{k}\right)e^{kx} + c$

C. $\frac{1}{k}\int e^{kx}\,dx$

D. $\frac{1}{k}\left(xe^{kx} - \int e^{kx}\,dx\right) + c$

E. $\frac{1}{k^2}\left(xe^{kx} - e^{kx}\right) + c$

$\frac{d}{dx}(xe^{kx}) = (kx+1)e^{kx}$

$\therefore xe^{kx} = \int (kx+1)e^{kx}dx$

$\therefore xe^{kx} = \int kxe^{kx}dx + \int e^{kx}dx$

$xe^{kx} - \int e^{kx}dx = \int kxe^{kx}dx$

$\therefore \frac{1}{k}\left(e^{kx} - \int e^{kx}dx\right) = \int xe^{kx}dx$

Question 10

For the curve $y = x^2 - 5$, the tangent to the curve will be parallel to the line connecting the positive x-intercept and the y-intercept when x is equal to

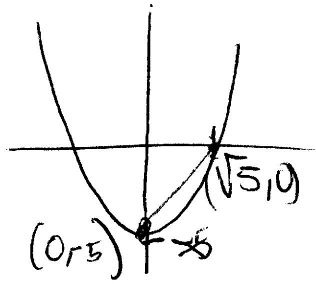
A. $\sqrt{5}$

B. 5

C. -5

D. $\dfrac{\sqrt{5}}{2}$

E. $\dfrac{1}{\sqrt{5}}$



Quick diagram!

$$m = \frac{0 - -5}{\sqrt{5} - 0} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$\therefore \frac{dy}{dx} = \sqrt{5}$$

$$\therefore 2x = \sqrt{5}$$

$$x = \frac{\sqrt{5}}{2}$$

Question 11

The function f has the property $f(x) - f(y) = (y - x) f(xy)$ for all non-zero real numbers x and y.
Which one of the following is a possible rule for the function?

~~**A.**~~ $f(x) = x^2$

~~**B.**~~ $f(x) = x^2 + x^4$

~~**C.**~~ $f(x) = x \log_e(x)$

D. $f(x) = \dfrac{1}{x}$

E. $f(x) = \dfrac{1}{x^2}$

$$f(x) = \frac{1}{x}$$

$$f(x) - f(y) = \frac{1}{x} - \frac{1}{y}$$

$$= \frac{y - x}{xy}$$

$$= (y - x) \times \frac{1}{xy}$$

$$= (y - x) f(xy)$$

Can't work for $f(x) = x^2$ Therefore B won't work. Also won't work for C, since $f(xy)$ would be converted into $f(x) + f(y)$

Question 12

The graph of a function f is obtained from the graph of the function g with rule $g(x) = \sqrt{2x - 5}$ by a reflection in the x-axis followed by a dilation from the y-axis by a factor of $\frac{1}{2}$.
Which one of the following is the rule for the function f?

A. $f(x) = \sqrt{5 - 4x}$

B. $f(x) = -\sqrt{x - 5}$

C. $f(x) = \sqrt{x + 5}$

D. $f(x) = -\sqrt{4x - 5}$

E. $f(x) = -\sqrt{4x - 10}$

$$g_1(x) = \sqrt{2x - 5}$$

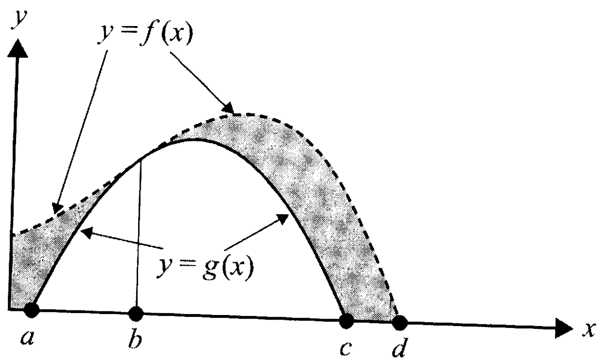
$$g_2(x) = -g_1(x) = -\sqrt{2x - 5}$$

$$g_3(x) = g_2(2x) = -\sqrt{2(2x) - 5}$$

$$= -\sqrt{4x - 5}$$

Question 13

Consider the graphs of the functions f and g shown below.



The area of the shaded region could be represented by

A. $\int_a^d (f(x) - g(x))\,dx$

B. $\int_0^d (f(x) - g(x))\,dx$

C. $\int_0^b (f(x) - g(x))\,dx + \int_b^c (f(x) - g(x))\,dx$

D. $\int_0^a f(x)\,dx + \int_a^c (f(x) - g(x))\,dx + \int_b^d f(x)\,dx$

E. $\int_0^d f(x)\,dx - \int_a^c g(x)\,dx$

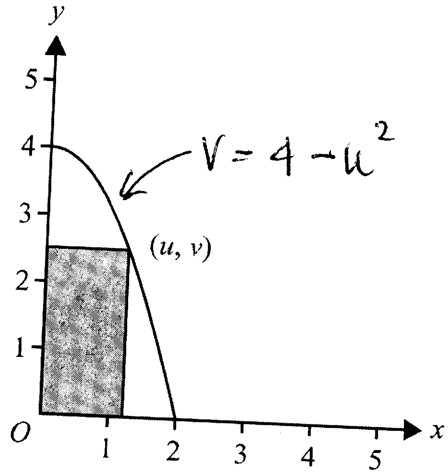
Grey area

= Area under f(x) over domain [0, d] — Area under g(x) on domain [a, c]

$= \int_0^d f(x)\,dx - \int_a^c g(x)\,dx$

Question 14

A rectangle is formed by using part of the coordinate axes and a point (u, v), where $u > 0$ on the parabola $y = 4 - x^2$.



$$\text{Area} = uv$$
$$= u(4-u^2)$$

Which one of the following is the maximum area of the rectangle?

A. 4

B. $\frac{2\sqrt{3}}{3}$

C. $\frac{8\sqrt{3}-4}{3}$

D. $\frac{8}{3}$

E. $\frac{16\sqrt{3}}{9}$

$$A(u) = 4u - u^3$$
$$A'(u) = 4 - 3u^2$$
$$= 0 \text{ if } u^2 = \frac{4}{3}$$
$$u = \frac{2}{\sqrt{3}}$$

$$A_{max} = A\left(\frac{2}{\sqrt{3}}\right) = \frac{4\times 2}{\sqrt{3}} - \left(\frac{2}{\sqrt{3}}\right)^3$$
$$= \frac{8}{\sqrt{3}} - \frac{8}{3\sqrt{3}} = \frac{16}{3\sqrt{3}} = \frac{16\sqrt{3}}{9}$$

Question 15

A box contains six red marbles and four blue marbles. Two marbles are drawn from the box, without replacement.

The probability that they are the same colour is

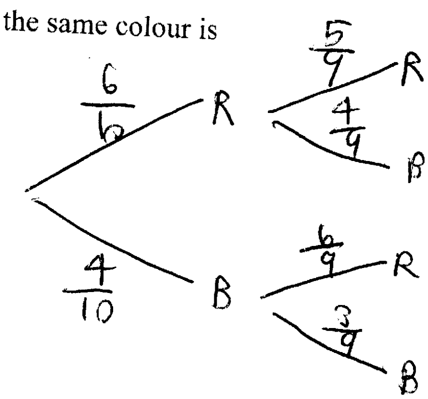
A. $\frac{1}{2}$

B. $\frac{28}{45}$

C. $\frac{7}{15}$

D. $\frac{3}{5}$

E. $\frac{1}{3}$



$$\Pr(RR) + \Pr(BB)$$
$$= \frac{6}{10}\times\frac{5}{9} + \frac{4}{10}\times\frac{3}{9}$$
$$= \frac{30}{90} + \frac{12}{90}$$
$$= \frac{42}{90} = \frac{7}{15}$$

Question 16

The random variable, X, has a normal distribution with mean 12 and standard deviation 0.25

If the random variable, Z, has the standard normal distribution, then the probability that X is greater than 12.5 is equal to

A. $\Pr(Z < -4)$

B. $\Pr(Z < -1.5)$

C. $\Pr(Z < 1)$

D. $\Pr(Z \geq 1.5)$

E. $\Pr(Z > 2)$

$X \stackrel{d}{=} N\left(\mu = 12, \sigma = \frac{1}{4}\right)$

$Z = \dfrac{X - 12}{\frac{1}{4}} = 4(X - 12)$

$\therefore \Pr(X > 12.5) = \Pr(Z > 2)$

Question 17

Inside a container there are one million coloured building blocks. It is known that 20% of the blocks are red. A sample of 16 blocks is taken from the container. For samples of 16 blocks, $\hat{P}$ is the random variable of the distribution of sample proportions of red blocks. (Do not use a normal approximation.)

$\Pr\left(\hat{P} \geq \frac{3}{16}\right)$ is closest to

A. 0.6482

B. 0.8593

C. 0.7543

D. 0.6542

E. 0.3211

$\hat{P} = \dfrac{X}{16}$ $\quad X \stackrel{d}{=} Bi\left(n = 16, p = \frac{1}{5}\right)$

$\Pr\left(\hat{P} \geq \frac{3}{16}\right)$

$= \Pr(X \geq 3) \simeq 0.6482$

Question 18

The continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} \dfrac{1}{4}\cos\left(\dfrac{x}{2}\right) & 3\pi \leq x \leq 5\pi \\ 0 & \text{elsewhere} \end{cases}$$

The value of a such that $\Pr(X < a) = \dfrac{\sqrt{3} + 2}{4}$ is

A. $\dfrac{19\pi}{6}$

B. $\dfrac{14\pi}{3}$

C. $\dfrac{10\pi}{3}$

D. $\dfrac{29\pi}{6}$

E. $\dfrac{17\pi}{3}$

$$\int_{3\pi}^{a} \frac{1}{4}\cos\left(\frac{x}{2}\right)dx = \frac{\sqrt{3}+2}{4}$$

Solving: $a = \dfrac{14\pi}{3}$

CAS TIP:

NOTE: When you type this in, you must specify domain of a $\quad \text{solve}\left(\int_{3\pi}^{a} \frac{1}{4}\cos\left(\frac{x}{2}\right)dx = \frac{\sqrt{3}+2}{4}, a\right)\Big|_{3\pi < a < 5\pi}$

Question 19

Consider the discrete probability distribution with random variable X shown in the table below.

x	-1	0	b	$2b$	4
$\Pr(X = x)$	a	b	b	$2b$	0.2

The smallest and largest possible values of $E(X)$ are respectively

A. -0.8 and 1

B. -0.8 and 1.6

C. 0 and 2.4

D. 0.2125 and 1

E. 0 and 1

$0 \le a \le 1$

$a + b + b + 2b + 0.2 = 1$

$\therefore 4b + a = 0.8$

$a = 0.8 - 4b$

$\therefore$ If $a = 0$, $b = 0.2$

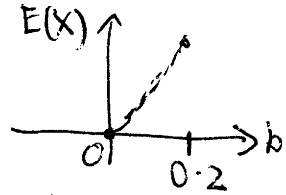
If $b = 0$, $a = 0.8$

$E(X) = -1 \times a + 0 \times b + b^2 + 4b^2 + 0.8$

$= -a + 5b^2 + 0.8$ But $a = 0.8 - 4b$

$\therefore E(X) = 4b - 0.8 + 5b^2 + 0.8$

$= 5b^2 + 4b$ where $0 \le b \le 0.2$



If $b = 0.2$,

$5 \times (0.2)^2 + 4 \times 0.2$

$= 0.2 + 0.8$

$= 1$

Question 20

Consider the transformation T, defined as

$$T : R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

The transformation T maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

If $\int_0^3 f(x)dx = 5$, then $\int_{-3}^0 g(x)dx$ is equal to

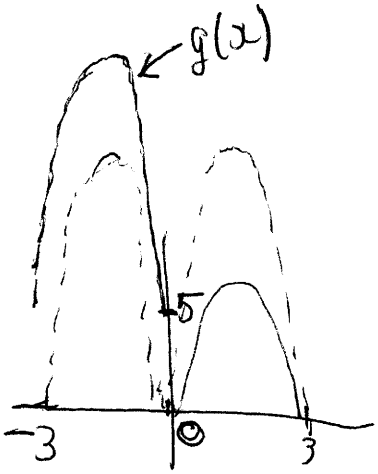
A. 0

B. 15

C. 20

D. 25

E. 30



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x \\ 3y + 5 \end{bmatrix}$$

- Reflection in y axis
- Dilation of factor 3 away from x-axis
- Translation of 5 units up

If A is original area,

new area $= 3 \times A + 5 \times 3$

$= 3A + 15$

$\therefore$ New area $= 3 \times 5 + 15 = 30$

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (11 marks)

Let $f:[0, 8\pi] \to R,\ f(x) = 2\cos\left(\frac{x}{2}\right) + \pi$.

a. Find the period and range of f. 2 marks

Period $= \frac{2\pi}{1/2} = 4\pi$

Range: $+2\uparrow$ $-2\downarrow$ $\begin{bmatrix} \pi+2 \\ \pi \\ \pi-2 \end{bmatrix}$ $[\pi-2, \pi+2]$

b. State the rule for the derivative function f'. 1 mark

$f'(x) = -\sin\left(\frac{x}{2}\right)$

c. Find the equation of the tangent to the graph of f at $x = \pi$. 1 mark

At $x=\pi$, $f'(\pi) = -\sin\left(\frac{\pi}{2}\right) = -1$

and $f(\pi) = 2\cos\left(\frac{\pi}{2}\right) + \pi = \pi$

$\therefore (x_1, y_1) = (\pi, \pi)$, $m = -1$

$y - \pi = -(x - \pi)$

$y - \pi = -x + \pi$

$y = -x + 2\pi$

Or simply use the Tangentline facility under Calculus menu

d. Find the equations of the tangents to the graph of $f:[0, 8\pi] \to R,\ f(x) = 2\cos\left(\frac{x}{2}\right) + \pi$ that have a gradient of 1. 2 marks

$\frac{dy}{dx} = 1 \quad \therefore \quad -\sin\left(\frac{x}{2}\right) = 1$

$\therefore \sin\left(\frac{x}{2}\right) = -1$

$\frac{x}{2} = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \quad \therefore x = 3\pi, 7\pi$

$f(3\pi) = \pi, \quad f(7\pi) = \pi$

$\therefore \quad y - \pi = 1(x - 3\pi) \qquad y - \pi = 1(x - 7\pi)$

$y = x - 2\pi \qquad y = x - 6\pi$

e. The rule of f' can be obtained from the rule of f under a transformation T, such that

$$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\pi \\ b \end{bmatrix}$$

Find the value of a and the value of b. 3 marks

$y = 2\cos\left(\frac{x}{2}\right) + \pi \quad \therefore \quad \frac{y - \pi}{2} = \cos\left(\frac{x}{2}\right)$

$y' = -\sin\left(\frac{x'}{2}\right) \quad \therefore \quad y' = \cos\left(\frac{\pi}{2} + \frac{x'}{2}\right)$

$y' = \frac{y - \pi}{2} \quad \therefore y' = \frac{y}{2} - \frac{\pi}{2}$

and $\frac{x}{2} = \frac{\pi + x'}{2}$

$\therefore x' = x - \pi$

$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\pi \\ -\frac{\pi}{2} \end{bmatrix} \quad \therefore a = \frac{1}{2}, \quad b = -\frac{\pi}{2}$

f. Find the values of x, $0 \le x \le 8\pi$, such that $f(x) = 2f'(x) + \pi$. 2 marks

$2\cos\left(\frac{x}{2}\right) + \pi = -2\sin\left(\frac{x}{2}\right) + \pi$

$\therefore \quad 2\cos\left(\frac{x}{2}\right) = -2\sin\left(\frac{x}{2}\right)$

$\therefore \tan\left(\frac{x}{2}\right) = -1, \quad \frac{x}{2} = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4},$

$x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}$

Question 2 (12 marks)

Consider the function $f(x) = -\frac{1}{3}(x+2)(x-1)^2$.

a. **i.** Given that $g'(x) = f(x)$ and $g(0) = 1$, show that $g(x) = -\frac{x^4}{12} + \frac{x^2}{2} - \frac{2x}{3} + 1$. 1 mark

Only worth 1 mark so not expected to show antidifferentiation by hand

$f(x) = -\frac{1}{3}(x+2)(x-1)^2$

$\therefore g'(x) = -\frac{1}{3}(x+2)(x-1)^2$

$\therefore g(x) = \int -\frac{1}{3}(x+2)(x-1)^2 \, dx$

Must show how to find the value of c.

$\therefore g(x) = -\frac{x^4}{12} + \frac{x^2}{2} - \frac{2x}{3} + c$

When $x=0$, $g(0) = 1$ $\therefore 1 = c$ $\therefore g(x) = -\frac{x^4}{12} + \frac{x^2}{2} - \frac{2x}{3} + 1$

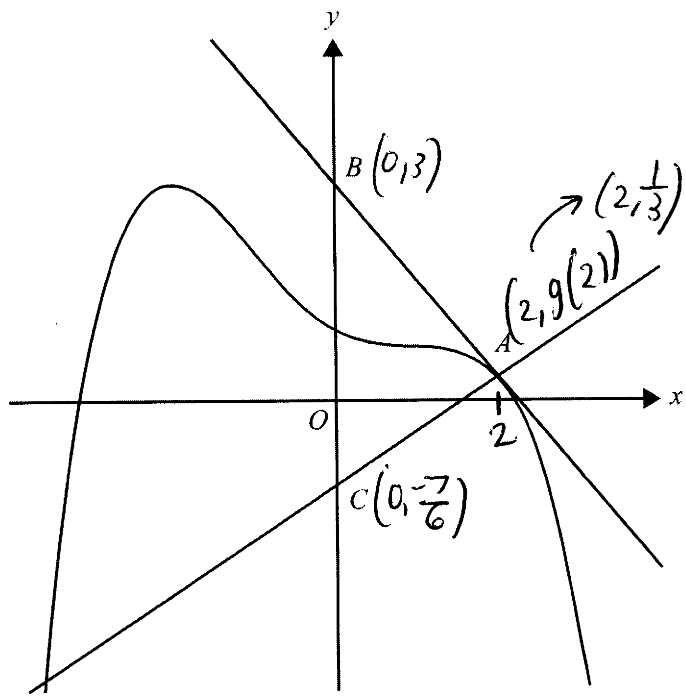
ii. Find the values of x for which the graph of $y = g(x)$ has a stationary point. 1 mark

$g'(x) = 0$

$\therefore x = -2, 1$

The diagram below shows part of the graph of $y = g(x)$, the tangent to the graph at $x = 2$ and a straight line drawn perpendicular to the tangent to the graph at $x = 2$. The equation of the tangent at the point A with coordinates $(2, g(2))$ is $y = 3 - \frac{4x}{3}$.

The tangent cuts the y-axis at B. The line perpendicular to the tangent cuts the y-axis at C.



$\overline{BA} = \sqrt{2^2 + \left(\frac{8}{3}\right)^2}$

$= \sqrt{4 + \frac{64}{9}} = \sqrt{\frac{100}{9}}$

$= \frac{10}{3}$

$\overline{AC} = \sqrt{(2-0)^2 + \left(\frac{1}{3} + \frac{7}{6}\right)^2}$

$= \sqrt{4 + \left(\frac{9}{6}\right)^2}$

$= \sqrt{4 + \frac{9}{4}}$

$= \sqrt{\frac{25}{4}}$

$= \frac{5}{2}$

b. **i.** Find the coordinates of B. 1 mark

$B = (0, 3)$

y-intercept of given tangent line.

ii. Find the equation of the line that passes through A and C and, hence, find the coordinates of C. 2 marks

For this line, $m_N = \frac{3}{4}$ (Perpendicular gradients)

$y - g(2) = \frac{3}{4}(x - 2)$

$y - \frac{1}{3} = \frac{3x}{4} - \frac{3}{2}$

$y = \frac{3x}{4} + \frac{2}{6} - \frac{9}{6}$

$y = \frac{3x}{4} - \frac{7}{6}$

$C = \left(0, -\frac{7}{6}\right)$

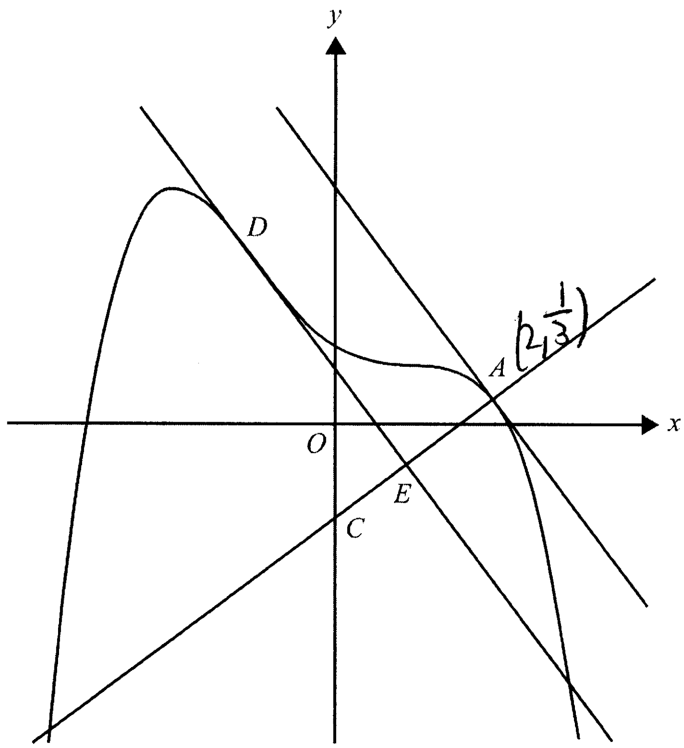
iii. Find the area of triangle ABC. 2 marks

Use right angle triangle formula for area.

$\text{Area} = \frac{1}{2} \times \overline{AC} \times \overline{BA}$

$= \frac{1}{2} \times \frac{5}{2} \times \frac{10}{3} = \frac{25}{6}$ sq. units

c. The tangent at D is parallel to the tangent at A. It intersects the line passing through A and C at E.



i. Find the coordinates of D. 2 marks

At D, $\frac{dy}{dx} = \frac{-4}{3}$

$\therefore x = -1, 2 \quad \therefore$ At D, $x = -1$

$g(-1) = \frac{25}{12}$

$\therefore D = \left(-1, \frac{25}{12}\right)$

ii. Find the length of AE. 3 marks

CAS Tip:
Use the
Tangentline
facility to
quickly get tangent
line equation at D.

Tangent at D: $y = -\frac{4x}{3} + \frac{3}{4}$

Solving: $y = \frac{-4x}{3} + \frac{3}{4}$ gives $E = \left(\frac{23}{25}, \frac{-143}{300}\right)$

$y = \frac{3}{4}x - \frac{7}{6}$

$\overline{AE} = \sqrt{\left(2 - \frac{23}{25}\right)^2 + \left(\frac{1}{3} + \frac{143}{300}\right)^2} = \sqrt{\frac{729}{400}} = \frac{27}{20}$

Question 3 (16 marks)

A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does **not** correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.

a. Determine the probability that at least one of the laptops is **not** correctly plugged into the trolley at the end of the lesson. Give your answer correct to four decimal places. 2 marks

X = no. of laptops not correctly plugged in

$X \overset{d}{=} Bi(n=22, p=0.1)$

$\Pr(X \geq 1) = 0.9015$

CAS
Use binomcdf

b. A teacher observes that at least one of the returned laptops is not correctly plugged into the trolley.

Given this, find the probability that fewer than five laptops are **not** correctly plugged in. Give your answer correct to four decimal places. 2 marks

$$\Pr(X < 5 \mid X \geq 1)$$

$$= \frac{\Pr(X < 5 \cap X \geq 1)}{\Pr(X \geq 1)}$$

$$= \frac{\Pr(1 \leq X \leq 4)}{\Pr(X \geq 1)}$$

$$= 0.9311$$

CAS

$$\frac{\text{binomcdf}(22,0.1,1,4)}{\text{binomcdf}(22,0.1,1,22)}$$

The time for which a laptop will work without recharging (the battery life) is normally distributed, with a mean of three hours and 10 minutes and standard deviation of six minutes. Suppose that the laptops remain out of the recharging trolley for three hours.

c. For any one laptop, find the probability that it will stop working by the end of these three hours. Give your answer correct to four decimal places. 2 marks

$$T \overset{d}{=} N(\mu = 190, \sigma = 6)$$

$$\Pr(T \leq 180) = 0.04779$$

$$\approx 0.0478$$

A supplier of laptops decides to take a sample of 100 new laptops from a number of different schools. For samples of size 100 from the population of laptops with a mean battery life of three hours and 10 minutes and standard deviation of six minutes, $\hat{P}$ is the random variable of the distribution of sample proportions of laptops with a battery life of less than three hours.

d. Find the probability that $\Pr(\hat{P} \geq 0.06 \mid \hat{P} \geq 0.05)$. Give your answer correct to three decimal places. Do not use a normal approximation. 3 marks

$$\hat{p} = \frac{x}{100}$$

$$\Pr(\hat{p} \geq 0.06 \mid \hat{p} \geq 0.05)$$

$$= \Pr(X \geq 6 \mid x \geq 5)$$

$$X \overset{d}{=} Bi(n = 100, p = 0.04779033)$$

$$= \frac{\Pr(x \geq 6)}{\Pr(x \geq 5)} \approx 0.6579 \approx 0.658$$

It is known that when laptops have been used regularly in a school for six months, their battery life is still normally distributed but the mean battery life drops to three hours. It is also known that only 12% of such laptops work for more than three hours and 10 minutes.

e. Find the standard deviation for the normal distribution that applies to the battery life of laptops that have been used regularly in a school for six months, correct to four decimal places. 2 marks

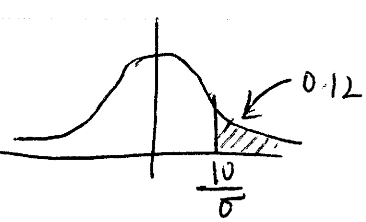
$$T \overset{d}{=} N(\mu = 180, \sigma = ?)$$

$$\Pr(T > 190) = 0.12 \qquad Z = \frac{T - 180}{\sigma}$$

$$\Pr\left(Z > \frac{10}{\sigma}\right) = 0.12$$

$$\therefore \frac{10}{\sigma} = \text{invNorm}(0.88, 0, 1)$$

$$\frac{10}{\sigma} = 1.1749868 \quad \therefore \sigma \approx 8.5107$$



The laptop supplier collects a sample of 100 laptops that have been used for six months from a number of different schools and tests their battery life. The laptop supplier wishes to estimate the proportion of such laptops with a battery life of less than three hours.

f. Suppose the supplier tests the battery life of the laptops one at a time.

Find the probability that the first laptop found to have a battery life of less than three hours is the third one. 1 mark

$T \stackrel{d}{=} N(\mu = 180, \sigma = 8.5109)$

$\therefore \Pr(T \leq 180) = 0.5$, $\Pr(T > 180) = 0.5$

$\xrightarrow{0.5} M \xrightarrow{0.5} M \xrightarrow{0.5} L$ $(0.5)^3 = 0.125$

The laptop supplier finds that, in a particular sample of 100 laptops, six of them have a battery life of less than three hours.

g. Determine the 95% confidence interval for the supplier's estimate of the proportion of interest. Give values correct to two decimal places. 1 mark

Using

1-prop z interval
$x = 6$
$n = 100$
Clevel $= 0.95$

CAS Only

Only worth 1 mark so fine to use the 1-prop z interval function on CAS

gives: $(0.01, 0.11)$

h. The supplier also provides laptops to businesses. The probability density function for battery life, x (in minutes), of a laptop after six months of use in a business is

$$f(x)=\begin{cases}\dfrac{(210-x)e^{\frac{x-210}{20}}}{400} & 0\le x\le 210\\ 0 & \text{elsewhere}\end{cases}$$

i. Find the **mean** battery life, in minutes, of a laptop with six months of business use, correct to two decimal places. 1 mark

$$E(X) = \int_0^{210} \frac{x(210-x)}{400} e^{\frac{x-210}{20}}\, dx$$

$$= 170.01 \text{ minutes}$$

ii. Find the **median** battery life, in minutes, of a laptop with six months of business use, correct to two decimal places. 2 marks

$$\int_0^{M} \frac{(210-x)e^{\frac{x-210}{20}}}{400}\, dx = \frac{1}{2}$$

Solving: $M = 176.45$ minutes

$(\text{since } 0 < M < 210)$

Question 4 (21 marks)

a. Express $\frac{2x+1}{x+2}$ in the form $a+\frac{b}{x+2}$, where a and b are non-zero integers. 2 marks

$$\begin{array}{r} 2 \\ x+2 \overline{)\,2x+1} \\ -(2x+4) \\ \hline -3 \end{array}$$

$$\frac{2x+1}{x+2} = 2 - \frac{3}{x+2}$$

b. Let $f: R\backslash\{-2\} \to R,\ f(x)=\frac{2x+1}{x+2}$.

i. Find the rule and domain of f^{-1}, the inverse function of f. 2 marks

dom(f) $\quad$ ran(f)

$\mathbb{R}\backslash\{-2\}$ $\quad$ $\mathbb{R}\backslash\{2\}$

dom(f^{-1}) $\quad$ ran(f^{-1})

$\mathbb{R}\backslash\{2\}$ $\quad$ $\mathbb{R}\backslash\{-2\}$

$\therefore$ dom(f^{-1}) $= \mathbb{R}\backslash\{2\}$

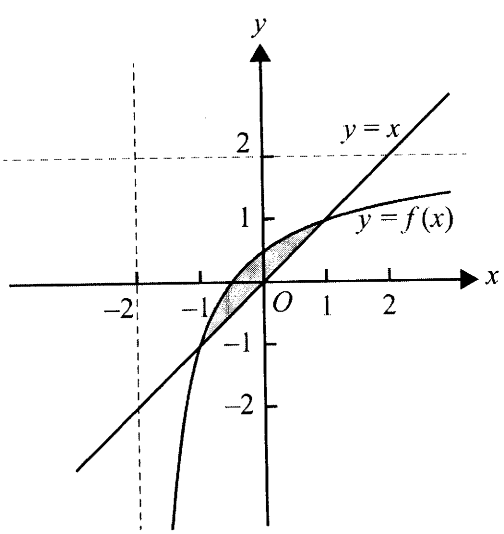
$$x = 2 - \frac{3}{y+2}$$

$$\therefore \frac{-3}{y+2} = x-2$$

$$y+2 = \frac{-3}{x-2}$$

$$\therefore f^{-1}(x) = \frac{-3}{x-2} - 2$$

ii. Part of the graphs of f and $y=x$ are shown in the diagram below.

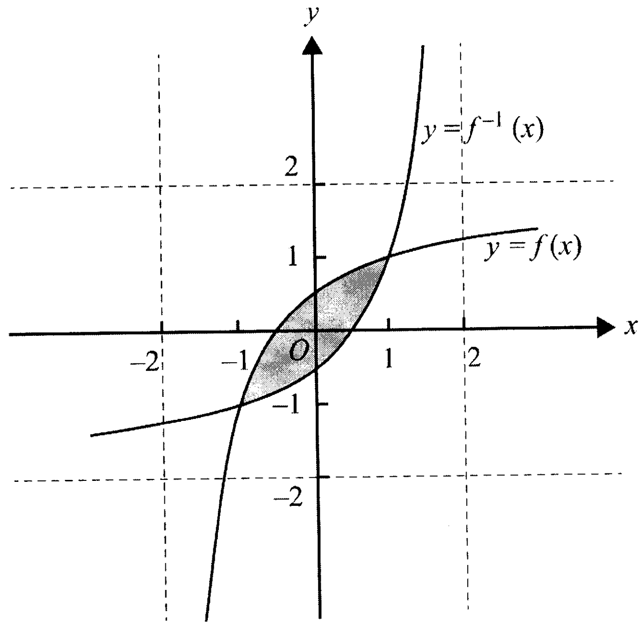


Find the area of the shaded region. 1 mark

$$\int_{-1}^{1}\left(\frac{2x+1}{x+2} - x\right)dx$$

$$= 4 - 3\log_e 3 \text{ sq. units}$$

iii. Part of the graphs of f and f^{-1} are shown in the diagram below.



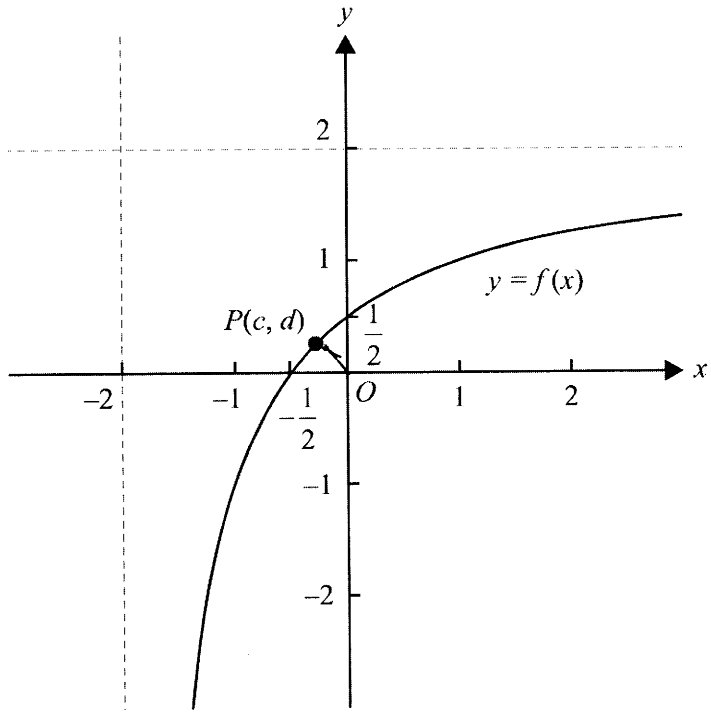
Find the area of the shaded region. 1 mark

$2(4 - 3\log_e 3)$

$= 8 - 6\log_e 3$ sq units

By the symmetry of inverse functions, this area will be double the previous area.

c. Part of the graph of f is shown in the diagram below.



The point $P(c, d)$ is on the graph of f.

Find the exact values of c and d such that the distance of this point to the origin is a minimum, and find this minimum distance. 3 marks

$d = f(c)$ $\therefore P = (c, f(c))$.

$O = (0, 0)$

$D(c) = \sqrt{c^2 + f(c)^2}$ where $f(x) = \frac{2x+1}{x+2}$

Solving: $D'(c) = 0$ gives

$c = -\sqrt{3} - 2, \ \sqrt{3} - 2$

Since $-\frac{1}{2} < c < 0$, $c = \sqrt{3} - 2$, $f(\sqrt{3}-2) = 2 - \sqrt{3}$

$D(\sqrt{3} - 2) = 2\sqrt{2} - \sqrt{6}$

Remember to state all the values required by the question

$\therefore c = \sqrt{3} - 2$, $d = 2 - \sqrt{3}$ and minimum distance $= 2\sqrt{2} - \sqrt{6}$

Let $g: (-k, \infty) \to R,\ g(x) = \dfrac{kx+1}{x+k}$, where $k > 1$.

d. Show that $x_1 < x_2$ implies that $g(x_1) < g(x_2)$, where $x_1 \in (-k, \infty)$ and $x_2 \in (-k, \infty)$. 2 marks

$$g(x_2) - g(x_1) = \frac{kx_2+1}{x_2+k} - \frac{kx_1+1}{x_1+k}$$

CAS TIP

Use Algebra/Polynomial Tools/CommonDenom to get this.

$$= \frac{k^2x_1 - k^2x_2 - x_1 + x_2}{(k+x_1)(k+x_2)}$$

Use Algebra/Factor to get the factorized from

$$= \frac{(k-1)(k+1)(x_2-x_1)}{(k+x_1)(k+x_2)}$$

Given: $k > 1$, $\Rightarrow (k-1) > 0$

$k+1 > 0$

Given $x_1 \in (-k, \infty)$, $x_1 + k > 0$

$x_2 \in (-k, \infty)$, $x_2 + k > 0$

$$\therefore g(x_2) - g(x_1) = \frac{(k-1)(k+1)}{(k+x_1)(k+x_2)}(x_2 - x_1)$$

where: $\dfrac{(k-1)(k+1)}{(k+x_2)(k+x_1)} > 0$

$\therefore g(x_2) - g(x_1) > 0$ if $x_2 - x_1 > 0$

$\therefore x_2 > x_1$ implies $g(x_2) > g(x_1)$

e. **i.** Let X be the point of intersection of the graphs of $y = g(x)$ and $y = -x$.

Find the coordinates of X in terms of k. 2 marks

$g(x) = \dfrac{kx+1}{x+k}$, $x \in (-k, \infty)$

$h(x) = -x$

Solving: $x = -k - \sqrt{k^2-1}$ or $\sqrt{k^2-1} - k$

Since $x \in (-k, \infty)$, $x = -k + \sqrt{k^2-1}$

$h(-k + \sqrt{k^2-1}) = k - \sqrt{k^2-1}$

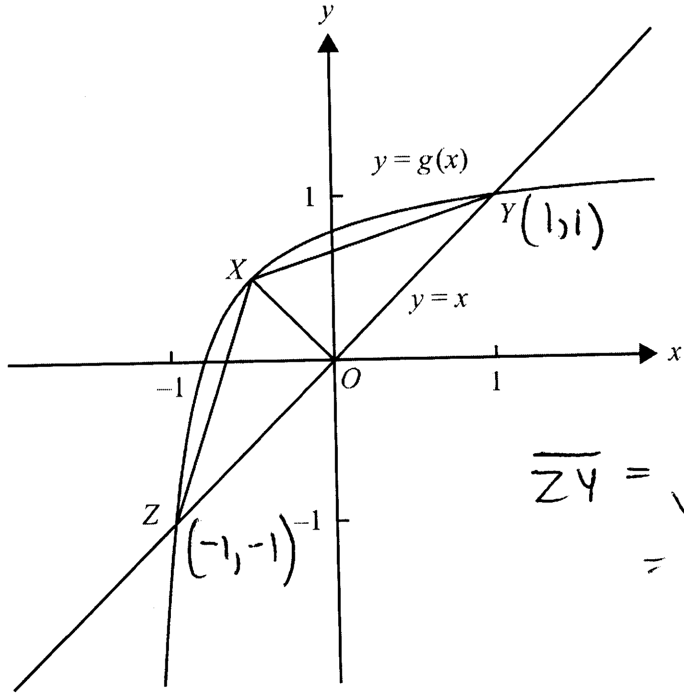
$\therefore X = \left(-k + \sqrt{k^2-1},\ k - \sqrt{k^2-1}\right)$

ii. Find the value of k for which the coordinates of X are $\left(-\frac{1}{2}, \frac{1}{2}\right)$. 2 marks

$\frac{1}{2} = k - \sqrt{k^2-1}$

Solving: $k = \frac{5}{4}$

iii. Let $Z(-1, -1)$, $Y(1, 1)$ and X be the vertices of the triangle XYZ. Let $s(k)$ be the square of the area of triangle XYZ.



$$\overline{ZY} = \sqrt{(-1-1)^2 + (-1-1)^2}$$
$$= \sqrt{4+4}$$
$$= \sqrt{8} = 2\sqrt{2}$$

Find the values of k such that $s(k) \geq 1$.

2 marks

$$s(k) = \left(\frac{1}{2}\,\overline{ZY}\;\overline{XO}\right)^2$$
$$= \frac{1}{4}\left(\sqrt{8}\right)^2\left(\overline{XO}\right)^2$$
$$= 2\left(\overline{XO}\right)^2$$

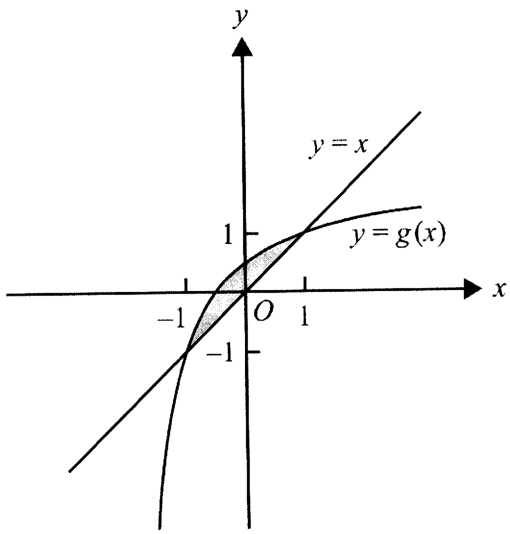
$$X = \left(-k+\sqrt{k^2-1},\ k-\sqrt{k^2-1}\right)$$

$$\overline{XO}^2 = \left(-k+\sqrt{k^2-1}\right)^2 + \left(k-\sqrt{k^2-1}\right)^2$$
$$= 2\left(k-\sqrt{k^2-1}\right)^2$$
$$\therefore\ s(k) = 4\left(k-\sqrt{k^2-1}\right)^2$$

Solving $s(k) \geq 1$ gives $1 \leq k \leq \frac{5}{4}$ or $k \leq -1$

But we are given: $k > 1$ $\quad \therefore\ 1 < k \leq \frac{5}{4}$

f. The graph of g and the line $y = x$ enclose a region of the plane. The region is shown shaded in the diagram below.



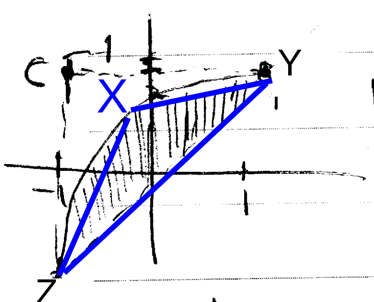
Let $A(k)$ be the rule of the function A that gives the area of this enclosed region. The domain of A is $(1, \infty)$.

i. Give the rule for $A(k)$. **2 marks**

$$A(k) = \int_{-1}^{1} \left(\frac{kx+1}{x+k} - x \right) dx$$

$$\therefore \quad A(k) = (k^2-1)\log_e\left(\frac{k-1}{k+1}\right) + 2k$$

ii. Show that $0 < A(k) < 2$ for all $k > 1$. **2 marks**



The required area is always contained within a right angled triangle ZCY of height 2 units and base 2 units

$\therefore \quad A(k) < \frac{1}{2} \times 2 \times 2 = 2$

Recall s(k) was the square of the area of triangleZXY

Also, $A(k) > \sqrt{s(k)}$ where $\sqrt{s(k)} = 2\left(k - \sqrt{k^2-1}\right)$ is the area of the triangle ZXY. Since $k - \sqrt{k^2-1} > 0$ for all k, $s(k) > 0$ $\quad \therefore 0 < A(k) < 2$

END OF QUESTION AND ANSWER BOOK