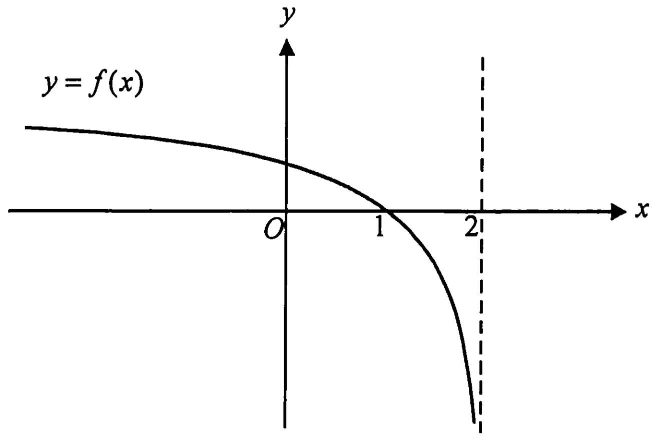
ANSWERS

Integrations revision -

Question 1 (17 marks)

Let $f:(-\infty,2)\to R,\ f(x)=\log_e(2-x)$.

The graph of f is shown below.



a. The graph of f is reflected in the y-axis to become the graph of g.

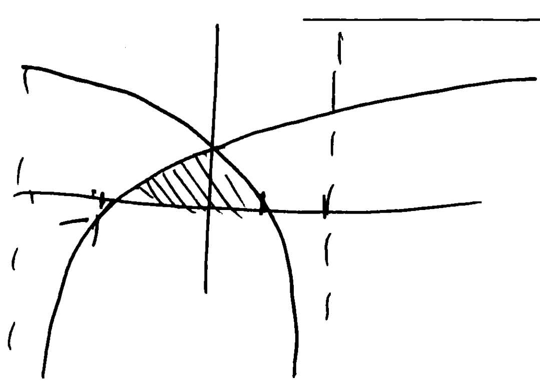
i. Find the rule and domain of g. 2 marks

$g(x) = f(-x) = \log_e(x+2)$

$x \in (-\infty, -2)$

ii. Find the area enclosed by the graphs of f and g and the x-axis 1 mark

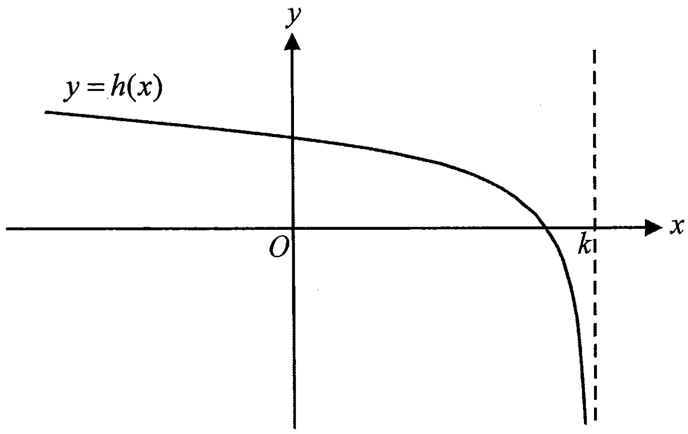
$$\int_{-1}^{0} \log_e(2+x)\,dx + \int_{0}^{1} \log_e(2-x)\,dx$$



$= 4\log_e 2 - 2$

Let $h:(-\infty,k)\to R,\ h(x)=\log_e(k-x)$, where $k>1$.

The graph of h is shown below.



b. **i.** Find, in terms of k, the coordinates of the x and y intercepts of the graph of h.

2 marks

Let $x=0$ $h(0)=\log_e k$ $\therefore$ y-int: $(0,\log_e k)$

Let $y=0$ $\therefore \log_e(k-x)=0$

$\therefore e^0=k-x$

$x=k-1$ x-int: $(k-1,0)$

ii. Show that if $h(x_1)>h(x_2)$ then $x_1<x_2$, where $x_1\in(-\infty,k)$ and $x_2\in(-\infty,k)$

2 marks

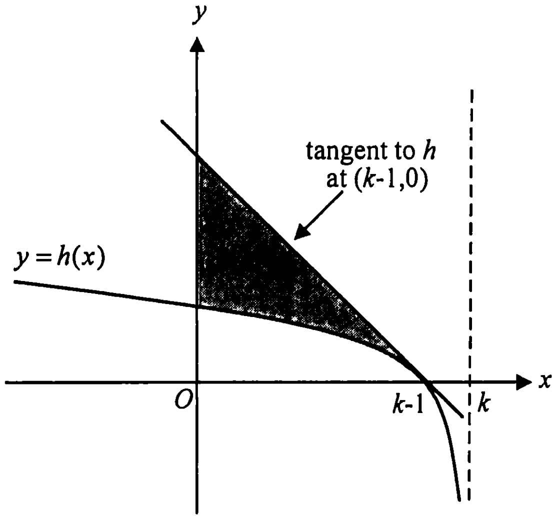
See p.9

iii. Find the equation of the tangent to h at the point $(k-1,0)$.

1 mark

$y=-x+k-1$

iv. Part of the graph of h and the tangent to h at the point $(k-1,0)$, are shown in the diagram below.



The area of the shaded region is given by the function A and the domain of A is $k>1$.

Show that the rule, $A(k)$ of this function is given by $A(k)=\frac{k^2}{2}-\frac{1}{2}-k\log_e(k)$. 2 marks

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v. Find the value of k for which the shaded area from part **iv.** equals the area enclosed by the graph of h and the x and y axes. Give your answer correct to 2 decimal places. 3 marks

See p.10

$$A(k) = \int_0^{k-1}\left(-x+k-1-\log_e(k-x)\right)dx$$

$$= \left[-\frac{x^2}{2}+(k-1)x\right]_0^{k-1} - \int_0^{k-1}\log_e(k-x)\,dx$$

$$= \frac{(k-1)^2}{2} - \left(k\log_e k - k + 1\right)$$

$$= \frac{k^2}{2} - \frac{1}{2} - k\log_e k$$

c. **i.** Find the rule and domain of h^{-1}, the inverse function of h. 2 marks

dom(h)	ran(h)
$(-\infty, k)$	R
dom(h^{-1})	ran(h^{-1})
R	$(-\infty, k)$

$x = \log_e(k-y)$

$e^x = k - y$

$y = k - e^x$

$h^{-1}: R \to R,\ h^{-1}(x) = k - e^x$

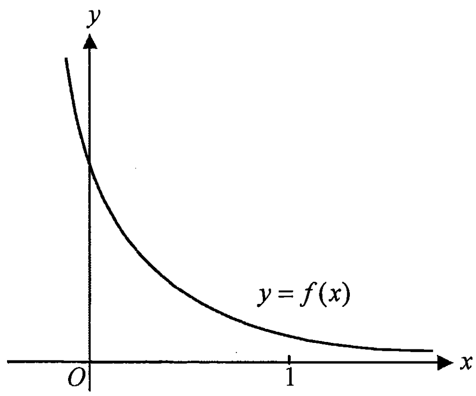
ii. The tangent to h at the point $(k-1,0)$ found in part **b. iii.** is also a tangent to h^{-1} at the point (p,q). Find the values of *p* and *q*. 2 marks

See p. 11

Question 2 (5 marks)

Let $f: R \to R, f(x) = e^{-2x}$.

Part of the graph of *f* is shown below.



a. Find the area enclosed by the graph of $y = f(x)$, the *x* and *y* axes and the line $x = 1$. 2 marks

$$\int_0^1 e^{-2x}\,dx = \left[-\frac{1}{2}e^{-2x}\right]_0^1$$

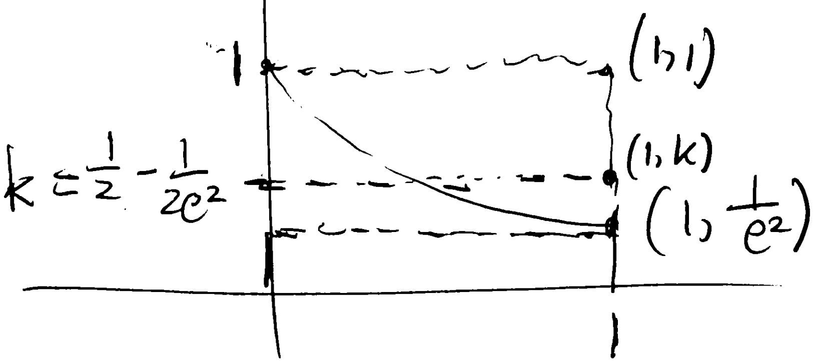
$$= -\frac{1}{2}e^{-2} + \frac{1}{2}$$

$$= \frac{1}{2} - \frac{1}{2e^2} \text{ sq. units}$$

b. Let k equal the average value of f between $x=0$ and $x=1$. Find k. 1 mark

$$k = \frac{1}{1-0}\int_0^1 e^{-2x}\,dx = \frac{1}{2} - \frac{1}{2e^2}$$

c. By using the graph and considering the area of appropriate rectangles, explain why $\frac{1}{e^2} < k < 1$. 2 marks

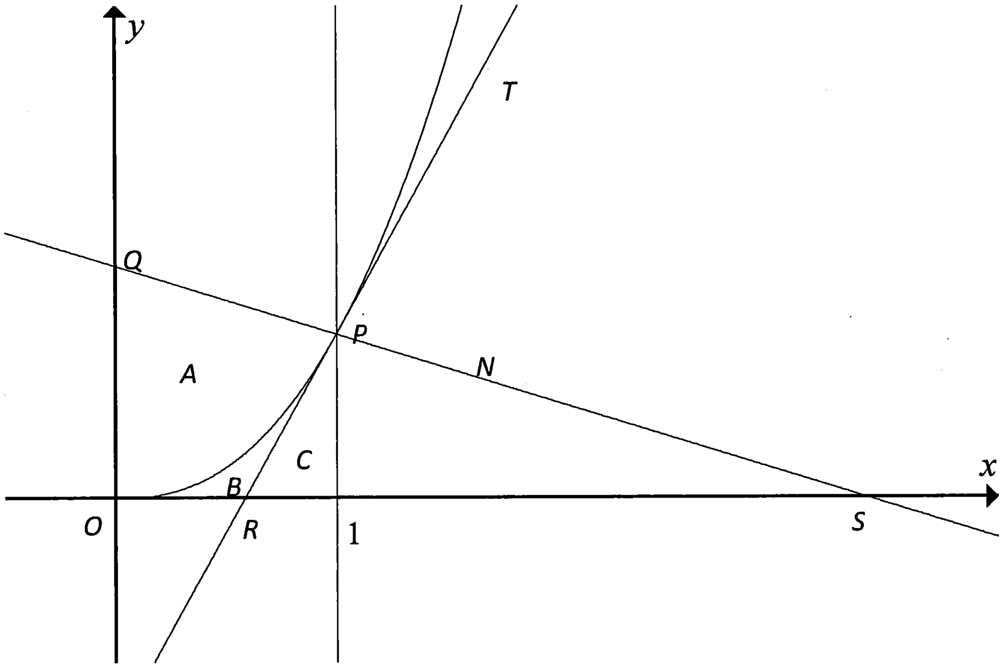


$$1 \times \frac{1}{e^2} < 1 \times k < 1 \times 1$$

$$\therefore \frac{1}{e^2} < k < 1$$

Question 3 (12 marks)

The diagram shows the point P, where $x = 1$ on part of the graph of $y = x^p$, where $p \in R$ and $p > 1$.



a. Let *T* be the tangent to the curve $y = x^p$ at the point *P*.

i. Find the equation of the tangent *T*.

1 mark

$y = px - p + 1$

ii. The tangent *T*, crosses the *x*-axis at the point *R*. Show that *R* has coordinates $\left(\frac{p-1}{p}, 0\right)$.

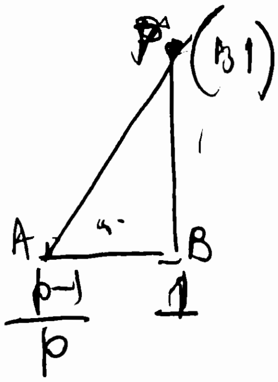
1 mark

Let $y = 0$ $\therefore$ $px - p + 1 = 0$

$x = \frac{p-1}{p}$

$\therefore \left(\frac{p-1}{p}, 0\right)$

iii. Let C be the area bounded by the triangular region, the tangent T, the points R and P, the x-axis and the line through $x = 1$. Find the area C in terms of p.



$$\text{Area} = \frac{AB \times 1}{2} \qquad \overline{AB} = 1 - \left(\frac{p-1}{p}\right) = \frac{1}{p}$$

1 mark

$$\therefore \text{Area} = \frac{1}{2p}$$

iv. The area B, is the area bounded by the curve $y = x^p$, the origin O, the tangent T and the x-axis. Hence or otherwise, write down using a definite integral the area B in terms of p.

1 mark

$$\text{Area } B + \frac{1}{2p} = \int_0^1 x^p \, dx$$

$$\therefore \text{Area } B = \int_0^1 x^p \, dx - \frac{1}{2p}.$$

v. Find the area B in terms of p.

1 mark

$$\left[\frac{x^{p+1}}{p+1}\right]_0^1 - \frac{1}{2p} = \frac{1}{p+1} - \frac{1}{2p}$$

vi. Find the value of p which maximizes the area B.

2 marks

$$A(p) = \frac{1}{p+1} - \frac{1}{2p}, \quad p > 1$$

Let $A'(p) = 0$

$$\therefore p = 1 + \sqrt{2}$$

Question 4 **(4 marks)**

a. If $f(x)=\frac{3\cos(2x)}{\sin(2x)}$ show that: $f'(x)=\frac{-6}{\sin^2(2x)}$.

See p.12

Let $u=3\cos(2x)$ $\quad v=\sin 2x$

$u'=-6\sin 2x$ $\quad v'=2\cos 2x$

$$\frac{dy}{dx}=\frac{-6\sin 2x\times\sin 2x-6\cos 2x\cos 2x}{(\sin 2x)^2}=\frac{-6(\sin^2 2x+\cos^2 2x)}{(\sin 2x)^2}=\frac{-6}{\sin^2 2x}$$

b. Hence, find: $\int\left(\frac{2}{\sin^2(2x)}+1\right)dx$.

$$\frac{d}{dx}\left(\frac{3\cos 2x}{\sin 2x}\right)=\frac{-6}{\sin^2 2x}$$

$$\therefore \frac{3\cos 2x}{\sin 2x}=\int\frac{-6}{\sin^2 2x}\,dx$$

$$\frac{-\cos 2x}{\sin 2x}=\int\frac{2}{\sin^2 2x}\,dx$$

$$\boxed{\therefore \int\left(\frac{2}{\sin^2 2x}+1\right)dx=\frac{-\cos 2x}{\sin 2x}+x+c}$$

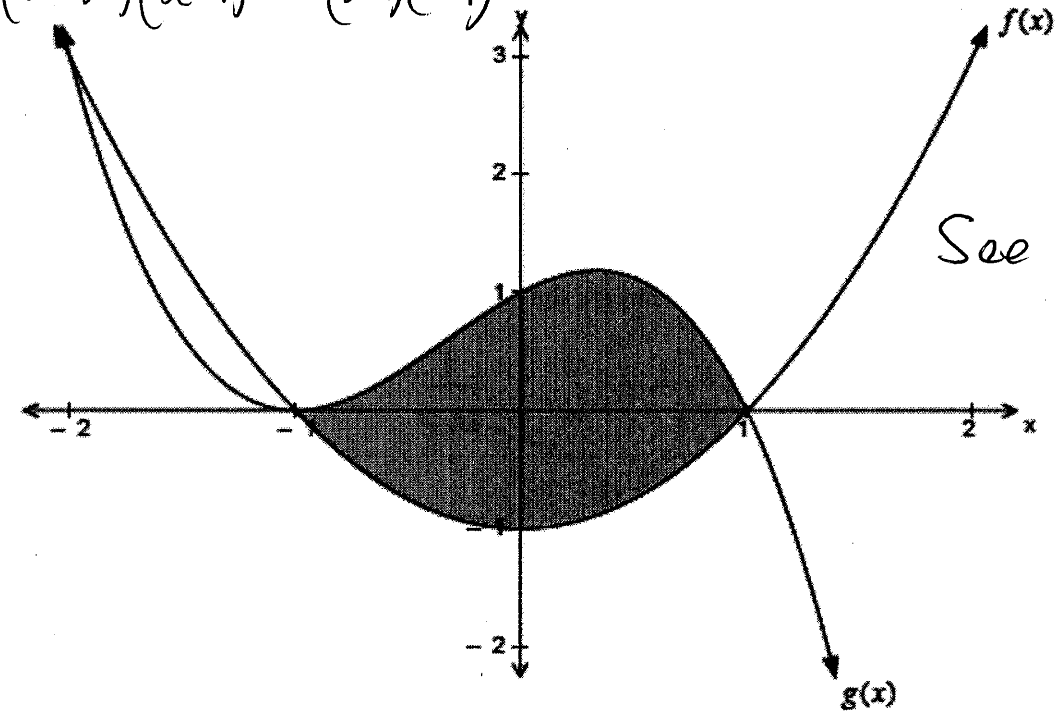
Question 5

Question 5

The graphs of $f(x)=(x-1)(x+1)$ and $g(x)=(1-x)(x+1)^2$ are shown below.

$$\int_{-1}^{1}(1-x)(x+1)^2-(x-1)(x+1)\,dx$$

$$\int g(x)-f(x)\,dx$$



See p.13

Find the exact area enclosed by the curves $f(x)$ and $g(x)$ for $x\in[-1,1]$.

Q.1(b)(ii)

$h(x_1) > h(x_2)$

$\therefore \log_e(k - x_1) > \log_e(k - x_2)$

$\therefore \log_e(k - x_1) - \log_e(k - x_2) > 0$

$\therefore \log_e\left(\dfrac{k - x_1}{k - x_2}\right) > 0$

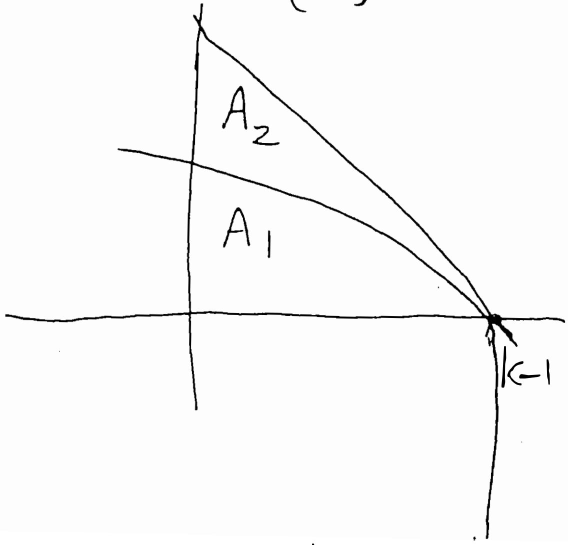
$\therefore \dfrac{k - x_1}{k - x_2} > 1$

$\therefore k - x_1 > k - x_2$ since $x_2 < k$

$\therefore -x_1 > -x_2$

$\therefore x_1 < x_2$

Q1 (b) (v)



$A_2 = A_1$

$$A_1 = \int_0^{k-1} h(x)\,dx$$

$$A_2 = \int_0^{k-1} \left(-x + k - 1 - h(x)\right) dx$$

$$\therefore \int_0^{k-1} h(x)\,dx = \int_0^{k-1} \left(-x+k-1-h(x)\right) dx$$

$$\therefore 2\int_0^{k-1} h(x)\,dx = \int_0^{k-1} (-x+k-1)\,dx$$

$$\int_0^{k-1} h(x)\,dx = \frac{1}{2} \times \frac{(k-1)^2}{2}$$

$$\therefore k\log_e k - k + 1 = \frac{(k-1)^2}{4}$$

Solving: $k = 1$ or $k = 5.1156609$

$\therefore$ But $k > 1$ $\quad \therefore k = 5.116$ = 5.12 (to 2 decimal places)

10

(c) (ii) $y = -x + k - 1$ is tangent to

$h^{-1}(x) = k - e^x$ at (p, q)

$\therefore h^{-1\prime}(p) = -1$

$h^{-1\prime}(x) = -e^x$

$\therefore -e^p = -1$

$\therefore e^p = 1$

$\therefore p = 0$

When $x = 0$, $h^{-1}(0) = k - e^0 = k - 1$

and on the tangent $y = -x + k - 1$

when $x = 0$, $y = k - 1$

$\therefore q = k - 1$

Q 4.

Let $u = 3\cos(2x)$ $\qquad v = \sin 2x$

$u' = -6\sin 2x \qquad v' = 2\cos 2x$

$$f'(x) = \frac{vu' - uv'}{v^2} = \frac{-6\sin 2x \times \sin 2x - 6\cos^2 2x}{(\sin 2x)^2}$$

$$= \frac{-6\sin^2 2x - 6\cos^2 2x}{(\sin 2x)^2}$$

$$= \frac{-6(\sin^2 2x + \cos^2 2x)}{\sin^2 2x}$$

$$= \frac{-6}{\sin^2 2x}$$

$$\frac{d}{dx}\left(\frac{3\cos 2x}{\sin 2x}\right) = \frac{-6}{\sin^2 2x}$$

$$\therefore \frac{d}{dx}\left(\frac{-\cos 2x}{\sin 2x}\right) = \frac{2}{\sin^2 2x}$$

$$\therefore \frac{-\cos 2x}{\sin 2x} = \int \frac{2}{\sin^2 2x}\, dx$$

$$\therefore \int \left(\frac{2}{\sin^2 2x} + 1\right) dx = \frac{-\cos 2x}{\sin 2x} + x + c$$

(12)

Q5.

$$\text{Area} = \int_{-1}^{1} g(x) - f(x) \, dx$$

$$= \int_{-1}^{1} (1-x)(x+1)^2 - (x-1)(x+1) \, dx$$

$$= \int_{-1}^{1} \left((1-x)(x+1)^2 + (1-x)(x+1) \right) dx$$

$$= \int_{-1}^{1} (1-x)(x+1)(x+1+1) \, dx$$

$$= \int_{-1}^{1} (1-x^2)(x+2) \, dx$$

$$= \int_{-1}^{1} \left(x+2-x^3-2x^2 \right) dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^4}{4} - \frac{2x^3}{3} \right]_{-1}^{1}$$

$$= \left(\frac{1}{2} + 2 - \frac{1}{4} - \frac{2}{3} \right) - \left(\frac{1}{2} - 2 - \frac{1}{4} + \frac{2}{3} \right)$$

$$= 2 + 2 - \frac{4}{3} = \frac{8}{3} \text{ sq. units}$$