ANSWERS.

QUESTION 16

According to agricultural scientists, a mathematical model can be used to approximate the production of cherries during a cherry season.

Cherry trees produce an average of 90 kilograms of cherries per tree. For the purposes of the model, this production weight can be considered to be constant for all trees at all times during the cherry season.

Before the cherry season, all cherries are unripe. The season starts with an off-peak period that lasts for 32 days. Every day during this period all cherries are inspected and:

- 98.5% are classified as Unripe and are left on the tree to be checked the next day
- 1.3% are classified as Ripe and are picked and sold
- 0.2% are classified as Damaged and are picked and made into jam.

(a) (i) After Day 1 of the off-peak period, how many kilograms of cherries per tree will be Unripe(*UR*), Ripe(*R*), and Damaged (*D*) according to this model?

UR: $0.985 \times 90 = 88.65$ kg
R: $0.013 \times 90 = 1.17$ kg
D: $0.002 \times 90 = 0.18$ kg

(1 mark)

(ii) Show that, after Day 2 of the off-peak period, each cherry tree will have produced 2.322 kilograms of Ripe cherries, according to this model.

$88.65 \times 0.985 + 1.17 = 2.322$ kg

(2 marks)

(b) Let $C_0 = \begin{array}{c} \begin{bmatrix} 90 \\ 0 \\ 0 \end{bmatrix} \end{array}\begin{array}{l} UR \\ R \\ D \end{array}$ and $T = \begin{array}{cl} \begin{array}{ccc} & \textit{today} & \\ UR & R & D \end{array} & \\ \begin{bmatrix} 0.985 & 0 & 0 \\ 0.013 & 1 & 0 \\ 0.002 & 0 & 1 \end{bmatrix} & \begin{array}{l} UR \\ R \\ D \end{array} \textit{tomorrow} \end{array}$

where C_0 represents the initital number of kg of unripe, ripe and damaged cherries on a tree, and T is the transition matrix that describes how the proportions of unripe, ripe and damaged cherries change from day to day.

(c) What is the significance of the number 1 in the second row of T?

Once a cherry becomes ripe, it cannot move to a different status.

1 mark

(ii) Find TC

$$TC = \begin{bmatrix} 88.65 \\ 1.17 \\ 0 \end{bmatrix}$$

(1 mark)

(iii) Find T^3C

$$T^3C = \begin{bmatrix} 86.01 \\ 3.46 \\ 0.53 \end{bmatrix} \begin{matrix} UR \\ R \\ D \end{matrix}$$

(1 mark)

(iv) Interpret your answer to part (b)(iii).

This gives the no. of unripe, ripe, damaged cherries at the end of Day 3.

(1 mark)

(c) A cherry orchard has 800 trees and each tree has at the start 90 kg of unripe cherries and no ripe nor damaged cherries. Write down the column matrix D_0 which represents the initial state matrix for the **entire** orchard

$$D_0 = \begin{bmatrix} 72000 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} UR \\ R \\ D \end{matrix}$$

(1 mark)

(d) The off peak season has 32 days. Find, to the nearest whole number, the number of kilograms of:

i. ripe cherries

$$D_{32} = \begin{bmatrix} 0.985 & 0 & 0 \\ 0.013 & 1 & 0 \\ 0.002 & 0 & 1 \end{bmatrix}^{32} \begin{bmatrix} 72000 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 44390.69 \\ 23928.07 \\ 3681.24 \end{bmatrix}$$

$\therefore$ **23928** kg of ripe cherries

ii. damaged cherries

3681 kg of damaged cherries

(1mark)

that were produced by the orchard in the off peak season

(d) The off-peak period of the cherry season is followed by a peak period that lasts for 64 days. Every day during the peak period all cherries are inspected and:

- 95.2% are classified as Unripe and are left on the tree to be checked the next day
- 4% are classified as Ripe and are picked and sold
- 0.8% are classified as Damaged and are picked and made into jam.

(i) Write down the new transition matrix U which would apply in the peak season.

$$U = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{ccc} UR & R & D \end{array}$$

$$U = \begin{bmatrix} 0.952 & 0 & 0 \\ 0.040 & 1 & 0 \\ 0.008 & 0 & 1 \end{bmatrix} \begin{array}{l} UR \\ R \\ D \end{array}$$

(1 mark)

NOTE: Must use the matrix found at the end of 32 days in off peak season from previous section.

ii. Determine, correct to the nearest whole number, the number of kilograms of:

a) ripe cherries

$$D_{64} \begin{bmatrix} 0.952 & 0 & 0 \\ 0.04 & 1 & 0 \\ 0.008 & 0 & 1 \end{bmatrix}^{64} \begin{bmatrix} 44390.69 \\ 23928.07 \\ 3681.24 \end{bmatrix} = \begin{bmatrix} 1906 \\ 59332 \\ 10,762 \end{bmatrix} \begin{array}{l} UR \\ R \\ D \end{array}$$

59,332 kg of ripe cherries

2 marks

b) damaged cherries

10,762 kg of damaged cherries

that will have been produced over the entire cherry season by this orchard.

1 mark

Scientists are studying the changes in the number of female frogs living along a creek bed. The scientists have established the following information about how the female frog population changes over time.

- No female frogs live beyond the age of 3 years.
- The stages of the frogs' development are: tadpole, frogteen and adult
- Females less than a year old are immature and cannot reproduce
- Only 50% of all tadpoles reach the age of one year (when they become a frogteen)
- Frog teens produce an average of 1.2 female tadpoles in that year.
- Only 20% of frogteens reach the age of 2 (when they become adults)
- Adult frogs produce on average 2 female tadpoles in that year.

The scientists have represented this information in a matrix.

$$P = \begin{array}{c} \textit{This year} \\ \begin{array}{ccc} T & F & A \end{array} \\ \begin{bmatrix} 0 & 1.2 & 2 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} \end{array} \begin{array}{l} T \\ F \\ A \end{array} \quad \textit{Next year}$$

The number of tadpoles, frogteens and adults at the time of the study are represented by the matrix:

$$O = \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} \begin{array}{l} T \\ F \\ A \end{array}$$

a. Find the matrix product PO and explain what information it gives.

$$PO = \begin{bmatrix} 0 & 1.2 & 2 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 136 \\ 100 \\ 16 \end{bmatrix} \begin{array}{l} T \\ F \\ A \end{array}$$

Gives the number of tadpoles, frogteens and adults after 1 year.

2 marks

b. Determine how many adult frogs, to the nearest whole number, there will be after 5 years, according to his scientific model.

$$P^5 O = \begin{bmatrix} 103.36 \\ 59.2 \\ 12.16 \end{bmatrix} \begin{array}{l} T \\ F \\ A \end{array}$$

1 mark

$\therefore$ 12 adults

c. Calculate decrease in the total frog population along the creek bed between year 2 and year 3. Give your answer correct to the nearest whole number.

$$P^3 O = \begin{bmatrix} 121.6 \\ 76 \\ 13.6 \end{bmatrix} \quad \text{Total} = 211.2 \approx 211$$

$$P^2 O = \begin{bmatrix} 152 \\ 68 \\ 20 \end{bmatrix} \quad \text{Total} = 240$$

Decrease
$= 240 - 211$
$= 29$

2 marks

c. In the longrun, describe what happens to the frog population in the creek, according to this scientific model. Justify your answer with appropriate calculations

$$P^{70} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 0.17 \\ 0.09 \\ 0.02 \end{bmatrix} \qquad P^{71} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.09 \\ 0.02 \end{bmatrix}$$

$$P^{100} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 0.008 \\ 0.004 \\ 0.001 \end{bmatrix} \qquad P^{101} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 0.008 \\ 0.004 \\ 0.001 \end{bmatrix}$$

In the longrun, the entire population heads to extinction (zero population)

2 marks

d. If each frogteen produces on average *x* tadpoles (instead of 1.2) it is found that the total population of frogs remains constant at a value close to 300. Find the value of *x*, correct to one decimal place.

Clearly x must be greater than 1.2 (because the population does not head towards zero)

Use trial and error:

if $x = 1.3$:
$$\begin{bmatrix} 0 & 1.3 & 2 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}^{50} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 4.6 \\ 2.5 \\ 0.54 \end{bmatrix}$$

and
$$\begin{bmatrix} 0 & 1.3 & 2 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}^{51} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 4.32 \\ 2.3 \\ 0.5 \end{bmatrix}$$

So the total will not stabilize at 300.

2 marks

If $x = 1.4$
$$\begin{bmatrix} 0 & 1.4 & 2 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}^{50} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 15.8 \\ 8.3 \\ 1.7 \end{bmatrix}$$

If $x = 1.5$
$$\begin{bmatrix} 0 & 1.5 & 2 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}^{51} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 54 \\ 27 \\ 6 \end{bmatrix}$$

Note: two consecutive calculations required to show that a steady state has been reached.

If $x = 1.6$
$$\begin{bmatrix} 0 & 1.6 & 2 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}^{52} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 182 \\ 91 \\ 18 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1.6 & 2 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}^{53} \begin{bmatrix} 200 \\ 80 \\ 20 \end{bmatrix} = \begin{bmatrix} 182 \\ 91 \\ 18 \end{bmatrix}$$

$\therefore$ Total stabilizes at $291 \approx 300$

$\therefore x = 1.6$

QUESTION 14

Researchers studying a colony of Tasmanian Devils in captivity have observed the following:

- 50% of female newborns survive to be 1 year old
- 50% of female 1-year-olds survive to be 2 years old
- 10% of female 2-year-olds survive to be 3 years old.

Each female aged 2 or 3 years gives birth to two female offspring at the same time, once a year.

Female Tasmanian Devils survive for 4 years, at most.

Source: © Arrxxx/Dreamstime.com

The yearly cycle of female Tasmanian Devils in this colony has been summarised in the following matrix:

$$L = \begin{array}{cl} \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix} & \begin{array}{l} \text{Breeding rate} \\ \text{Survival of newborns} \\ \text{Survival of 1-year-olds} \\ \text{Survival of 2-year-olds} \end{array} \end{array}$$

The column matrix F_0 represents the initial female population of the colony:

$$F_0 = \begin{array}{cl} \begin{bmatrix} 40 \\ 22 \\ 10 \\ 5 \end{bmatrix} & \begin{array}{l} \text{Newborns} \\ \text{1-year-olds} \\ \text{2-year-olds} \\ \text{3-year-olds} \end{array} \end{array}$$

(a) How many female Tasmanian Devils were there initially?

$40 + 22 + 10 + 5 = 77$

(1 mark)

(b) How many 1-year-old females will survive to be 2 years old?

$0.5 \times 22 = 11$

(1 mark)

(c) (i) Calculate the matrix $F_1 = LF_0$.

$$F_1 = \begin{bmatrix} 30 \\ 20 \\ 11 \\ 1 \end{bmatrix}$$

(1 mark)

(ii) Find the total female Tasmanian Devil population represented by $F_1 = LF_0$.

62

(1 mark)

(iii) Determine the percentage change in the female Tasmanian Devil population in part (c)(ii) compared with the initial colony.

$77 - 62 = 15$

$$\frac{15}{77} \times 100\% = 19.48\%$$

(2 marks)

(d) (i) Calculate $F_2 = L^2 F_0$.

$$F_2 = \begin{bmatrix} 24 \\ 15 \\ 10 \\ 11 \end{bmatrix}$$

(1 mark)

(ii) What does the matrix F_2 represent?

The female population after 2 years

(2 marks)

In another colony of the same species of Tasmanian Devil, the number of females in the colony can be represented by:

$$F_1 = \begin{bmatrix} 38 \\ 5 \\ 2 \\ 1 \end{bmatrix}.$$

(e) Solve for a, b, c, and d if $LF_0 = F_1$ and:

$$L = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix} \text{ and } F_0 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 38 \\ 5 \\ 2 \\ 1 \end{bmatrix}$$

Using:
$X = A^{-1} . K$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 38 \\ 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 10 \\ 9 \end{bmatrix}$$

(4 marks)

∴ There were initially 9 3 year old Devils.

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