ANSWERS

2015 FURMATH EXAM 2

Extended Response Revision Questions
AUGUST 30 LESSON

Module 6: Matrices

Question 1 (5 marks)

Students in a music school are classified according to three ability levels: beginner (B), intermediate (I) or advanced (A).

Matrix S_0, shown below, lists the number of students at each level in the school for a particular week.

$$S_0 = \begin{array}{c} \begin{bmatrix} 20 \\ 60 \\ 40 \end{bmatrix} \end{array} \begin{array}{l} B \\ I \\ A \end{array}$$

a. How many students in total are in the music school that week? 1 mark

20 + 60 + 40 = 120

The music school has four teachers, David (D), Edith (E), Flavio (F) and Geoff (G).

Each teacher will teach a proportion of the students from each level, as shown in matrix P below.

$$P = \begin{array}{c} \begin{array}{cccc} D & E & F & G \end{array} \\ \begin{bmatrix} 0.25 & 0.5 & 0.15 & 0.1 \end{bmatrix} \end{array}$$

The matrix product, $Q = S_0P$, can be used to find the number of students from each level taught by each teacher.

b. **i.** Complete matrix Q, shown below, by writing the missing elements in the shaded boxes. 1 mark

$$Q = \begin{array}{c} \begin{array}{cccc} D & E & F & G \end{array} \\ \begin{bmatrix} 5 & \boxed{10} & 3 & 2 \\ 15 & 30 & \boxed{9} & 6 \\ 10 & 20 & 6 & 4 \end{bmatrix} \end{array} \begin{array}{l} B \\ I \\ A \end{array}$$

ii. How many intermediate students does Edith teach? 1 mark

30

$$\begin{bmatrix} 20 \\ 60 \\ 40 \end{bmatrix} \begin{array}{l} B \\ I \\ A \end{array} \begin{array}{c} \begin{array}{cccc} D & E & F & G \end{array} \\ \begin{bmatrix} 0.25 & 0.5 & 0.15 & 0.1 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cccc} & E & & \end{array} \\ \begin{bmatrix} 20 \times 0.25 & 20 \times 0.5 & 20 \times 0.15 & 20 \times 0.1 \\ 60 \times 0.25 & 60 \times 0.5 & \cdots & \cdots \end{bmatrix} \end{array}$$

The music school pays the teachers \$15 per week for each beginner student, \$25 per week for each intermediate student and \$40 per week for each advanced student.

These amounts are shown in matrix C below.

1x3
C

$$C = \begin{array}{c} \begin{array}{ccc} B & I & A \end{array} \\ \begin{bmatrix} 15 & 25 & 40 \end{bmatrix} \end{array}$$

The amount paid to each teacher each week can be found using a matrix calculation.

c. **i.** Write down a matrix calculation in terms of Q and C that results in a matrix that lists the amount paid to each teacher each week. 1 mark

$C \cdot Q$

ii. How much is paid to Geoff each week? 1 mark

$15 \times 2 + 25 \times 6 + 40 \times 4$

$= 30 + 150 + 160$

$= \$340$

To calculate David's payment:

$5 \times 15 + 15 \times 25 + 10 \times 40$

$\therefore$ Column of Q is multiplied by row of C

Therefore, we require:

$$\begin{bmatrix} 15 & 25 & 40 \end{bmatrix} \begin{array}{c} \begin{array}{cccc} D & E & F & G \end{array} \\ \begin{bmatrix} 5 & 10 & 3 & 2 \\ 15 & 30 & 9 & 6 \\ 10 & 20 & 6 & 4 \end{bmatrix} \end{array}$$

Question 2 (3 marks)

The ability level of the students is assessed regularly and classified as beginner (B), intermediate (I) or advanced (A).

After each assessment, students either stay at their current level or progress to a higher level.

Students cannot be assessed at a level that is lower than their current level.

The expected number of students at each level after each assessment can be determined using the transition matrix, T_1, shown below.

$$\begin{array}{c} \text{before assessment} \\ \begin{array}{cccc} & B & I & A \end{array} \end{array}$$

$$T_1 = \begin{bmatrix} 0.50 & 0 & 0 \\ 0.48 & 0.80 & 0 \\ 0.02 & 0.20 & 1 \end{bmatrix} \begin{array}{c} B \\ I \\ A \end{array} \quad \text{after assessment}$$

a. The element in the third row and third column of matrix T_1 is the number 1.

Explain what this tells you about the advanced-level students. 1 mark

The Advanced level students remain advanced after each assessment.

Let matrix S_n be a state matrix that lists the number of students at beginner, intermediate and advanced levels after n assessments.

The number of students in the school, immediately before the first assessment of the year, is shown in matrix S_0 below.

$$S_0 = \begin{bmatrix} 20 \\ 60 \\ 40 \end{bmatrix} \begin{array}{c} B \\ I \\ A \end{array}$$

b. **i.** Write down the matrix S_1 that contains the expected number of students at each level after one assessment.

Write the elements of this matrix correct to the nearest whole number. 1 mark

$S_1 = T_1 S_0$

$= \begin{bmatrix} 10 \\ 58 \\ 52 \end{bmatrix}$

ii. How many intermediate-level students have become advanced-level students after one assessment? 1 mark

$60 \times 0.2 = 12$

Question 3 (7 marks)

A new model for the number of students in the school after each assessment takes into account the number of students who are expected to leave the school after each assessment.

After each assessment, students are classified as beginner (B), intermediate (I), advanced (A) or left the school (L).

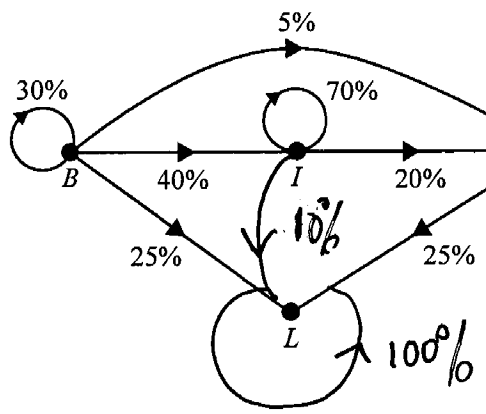
Let matrix T_2 be the transition matrix for this new model.

Matrix T_2, shown below, contains the percentages of students who are expected to change their ability level or leave the school after each assessment.

$$\begin{array}{c} \text{before assessment} \\ \begin{array}{cc} & \begin{array}{cccc} B & I & A & L \end{array} \\ T_2 = & \begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix} \begin{array}{c} B \\ I \\ A \\ L \end{array} \end{array} \end{array} \quad \text{after assessment}$$

a. An incomplete transition diagram for matrix T_2 is shown below.

Complete the transition diagram by adding the missing information. 2 marks



The number of students at each level, immediately before the first assessment of the year, is shown in matrix R_0 below.

$$R_0 = \begin{array}{c} \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} B \\ I \\ A \\ L \end{array}$$

Matrix T_2, repeated below, contains the percentages of students who are expected to change their ability level or leave the school after each assessment.

before assessment

$$T_2 = \begin{array}{c} \begin{array}{cccc} B & I & A & L \end{array} \\ \begin{bmatrix} 0.30 & 0 & 0 & 0 \\ 0.40 & 0.70 & 0 & 0 \\ 0.05 & 0.20 & 0.75 & 0 \\ 0.25 & 0.10 & 0.25 & 1 \end{bmatrix} \end{array} \begin{array}{l} B \\ I \\ A \\ L \end{array} \quad \text{after assessment}$$

b. What percentage of students is expected to leave the school after the first assessment? 1 mark

$0.25 \times 20 + 0.1 \times 60 + 0.25 \times 40 = 5 + 6 + 10 = 21$

$\frac{21}{120} \times 100\% = \frac{7}{40} \times 100\% = 17.5\%$

c. How many advanced-level students are expected to be in the school after two assessments? Write your answer correct to the nearest whole number. 1 mark

$T_2^2 R_0 = T_2^2 \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.8 \\ 37.4 \\ 42.55 \\ 38.25 \end{bmatrix}$

∴ 43 Advanced students

d. After how many assessments is the number of students in the school, correct to the nearest whole number, first expected to drop below 50? 1 mark

∴ After 5 assessments

$R_1 = \begin{bmatrix} 6 \\ 50 \\ 43 \\ 21 \end{bmatrix}$ $R_3 = \begin{bmatrix} 0.54 \\ 26.9 \\ 39.48 \\ 53.08 \end{bmatrix}$ $R_5 = \begin{bmatrix} 0.0486 \\ 13.397 \\ 30.08 \\ 76.47 \end{bmatrix}$

Total ≈ 43

$R_2 = \begin{bmatrix} 1.8 \\ 37.4 \\ 42.55 \\ 38.25 \end{bmatrix}$ $R_4 = \begin{bmatrix} 0.162 \\ 19.046 \\ 35.019 \\ 65.77 \end{bmatrix}$

Total ≈ 54

Another model for the number of students in the school after each assessment takes into account the number of students who are expected to join the school after each assessment.

Let R_n be the state matrix that contains the number of students in the school immediately after n assessments.

Let V be the matrix that contains the number of students who join the school after each assessment.

Matrix V is shown below.

$$V = \begin{array}{c} \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} B \\ I \\ A \\ L \end{array}$$

The expected number of students in the school after n assessments can be determined using the matrix equation

$$R_{n+1} = T_2 \times R_n + V$$

where

$$R_0 = \begin{array}{c} \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} B \\ I \\ A \\ L \end{array}$$

e. Consider the intermediate-level students expected to be in the school after three assessments.
How many are expected to become advanced-level students after the next assessment?
Write your answer correct to the nearest whole number. 2 marks

$$R_1 = T_2 \times \begin{bmatrix} 20 \\ 60 \\ 40 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 52 \\ 46 \\ 21 \end{bmatrix}$$

$$R_2 = T_2 \times \begin{bmatrix} 10 \\ 52 \\ 46 \\ 21 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 42.4 \\ 48.4 \\ 40.2 \end{bmatrix}$$

$$R_3 = T_2 \times \begin{bmatrix} 7 \\ 42.4 \\ 48.4 \\ 40.2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6.1 \\ 34.48 \\ 48.13 \\ 58.29 \end{bmatrix}$$

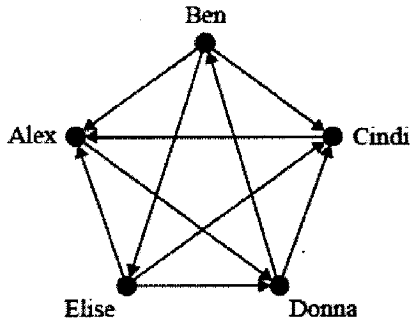
After 3 assessements, we have 34.48 Intermediate Students. The number expected to go into Advanced after next assessment $= 0.2 \times 34.48 = 6.896 \simeq 7$

$\therefore$ 7 students.

END OF QUESTION AND ANSWER BOOK

Question

The directed graph below shows the results of a chess competition between five players: Alex, Ben, Cindi, Donna and Elise.



Each arrow indicates the winner of individual games. For example, the arrow from Alex to Donna indicates that Alex beat Donna in their game.

The sum of their one step and two step dominances is calculated in order to rank the players.

a. Determine the one-step dominance matrix for this competition.

$$D = \begin{array}{c} \\ A \\ B\ \text{def} \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b. Determine the two step dominance matrix.

$$D^2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

c. Determine the matrix $T = D + D^2$ and hence rank the players.

$$T = D + D^2$$

$$= \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 2 & 0 \end{bmatrix} \begin{array}{c} \text{TOTAL} \\ 3 \\ 8 \\ 2 \\ 6 \\ 7 \end{array}$$

1st: B
2nd: E
3rd: D
4th: A
5th: C