The total cost of one ice-cream and three soft drinks at Catherine's shop is \$9.

The total cost of two ice-creams and five soft drinks is \$16.

Let x be the cost of an ice-cream and y be the cost of a soft drink.

The matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ is equal to

$x + 3y = 9$

$2x + 5y = 16$

A. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

B. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

D. $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

E. $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

$A. \quad X = K$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 16 \end{bmatrix}$$

$X = A^{-1}K$

$$X = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$$

$$X = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$$

Question 2

$$y - z = 8$$
$$5x - y = 0$$
$$x + z = 4$$

The system of three simultaneous linear equations above can be written in matrix form as

A. $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 5 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & -1 \\ 5 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

C. $\begin{bmatrix} 1 & -1 \\ 5 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 5 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 5 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

$0x + y - z = 8$

$5x - y + 0z = 0$

$x + 0y + z = 4$

$$\begin{bmatrix} 0 & 1 & -1 \\ 5 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$$

Question 4

$$2.8x + 0.7y = 10$$
$$1.4x + ky = 6$$

The set of simultaneous linear equations above does **not** have a solution if k equals

A. –0.35
B. –0.250
C. 0
D. 0.25
E. 0.35

$$\underset{A}{\begin{bmatrix} 2.8 & 0.7 \\ 1.4 & k \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{K}{\begin{bmatrix} 10 \\ 6 \end{bmatrix}}$$

For no unique solution, $\det(A) = 0$

$$\therefore 2.8k = 0.7 \times 1.4$$
$$k = \frac{0.7 \times 1.4}{2.8} = 0.35$$

Question 3

$$x + z = 6$$
$$2y + z = 8$$
$$2x + y + 2z = 15$$

The **solution** of the simultaneous equations above is given by

A.
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & -1 & 2 \\ -2 & 0 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$$

B.
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$$

C.
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$$

D.
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$$

E.
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$$

$$x + 0y + z = 6$$
$$0x + 2y + z = 8$$
$$2x + y + 2z = 15$$

$$\underset{A}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{K}{\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}}$$

$$X = A^{-1} . K$$

$$X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$$

$$X = \begin{bmatrix} -3 & -1 & 2 \\ -2 & 0 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$$

Question 3

Each of the following four matrix equations represents a system of simultaneous linear equations.

$A \cdot X = k$

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$ $\det(A) = 1\times 2 - 3\times 0 = 2 \neq 0$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$ $\det(A) = 1\times 2 - 2\times 1 = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$ $\det(A) = 1\times 2 - 0\times 0 = 2$

$$\begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$ $\det(A) = 0\times 2 - 3\times 0 = 0$

How many of these systems of simultaneous linear equations have a unique solution?

A. 0

B. 1

C. 2

D. 3

E. 4

Question 8

Consider the following matrix A.

$$A = \begin{bmatrix} 3 & k \\ -4 & -3 \end{bmatrix}$$

A is equal to its inverse A^{-1} for a particular value of k.

This value of k is

A. −4

B. −2

C. 0

D. 2

E. 4

Using CAS: if $A = \begin{bmatrix} 3 & k \\ -4 & -3 \end{bmatrix}$, then

$A^{-1} = \begin{bmatrix} \frac{-3}{4k-9} & \frac{-9}{4(4k-9)} - \frac{1}{4} \\ \frac{4}{4k-9} & \frac{3}{4k-9} \end{bmatrix}$

For A to be equal to A^{-1}, all the corresponding elements must be identical

$\therefore 3 = \frac{-3}{4k-9}$ $\therefore 4k-9 = -1$

$\therefore 1 = \frac{-1}{4k-9}$ $\therefore k = 2$

Question 4

The matrix equation $\begin{bmatrix} 4 & 2 & 8 \\ 2 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 6 \end{bmatrix}$ can be used to solve the system of simultaneous linear equations

A. $4x + 2y + 8z = 7$
$2x + 3y = 2$
$3x - y = 6$

$4x + 2y + 8z = 7$
$2x + 0y + 3z = 2$
$0x + 3y - z = 6$

$4x + 2y + 8z = 7$
$2x + 3z = 2$
$3y - z = 6$

B. $4x + 2y + 8z = 7$
$2x + 3y = 2$
$3y - z = 6$

C. $4x + 2y + 8z = 7$
$2y + 3z = 2$
$3x - z = 6$

D. $4x + 2y + 8z = 7$
$2x + 3z = 2$
$3y - z = 6$

E. $4x + 2y + 8z = 7$
$2x + 3z = 2$
$3x - z = 6$

Question 6

A. X = K

The solution of the matrix equation $\begin{bmatrix} 0 & -3 & 2 \\ 1 & 1 & 1 \\ -2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$ is

A. $\begin{bmatrix} 1 \\ 24 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

$X = \begin{bmatrix} 0 & -3 & 2 \\ 1 & 1 & 1 \\ -2 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$

C. $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

D. $\begin{bmatrix} -11 \\ \frac{4}{3} \\ 8 \end{bmatrix}$

$X = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

E. $\begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$

$x + 0.y = 0$
$x + y = 6$ $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$ $\therefore \det(A) = 1\times1 - 1\times0$
$= 1$

Question 7

How many of the following five sets of simultaneous linear equations have a unique solution?

$4x + 2y = 10$ $2x + y = 5$	$x = 0$ $x + y = 6$	$x - y = 3$ $x + y = 3$	$2x + y = 5$ $2x + y = 10$	$x = 8$ $y = 2$

$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$
$\det(A) = 4\times1 - 2\times2$
$= 0$

$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$
$\det(A) = 0$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$
$\det(A) = 1\times1 - 1\times-1$
$= 1 + 1$
$= 2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$
$\det(A) = 1\times1 - 0$
$= 1$

A. 1
B. 2
C. 3
D. 4
E. 5

EXTENDED RESPONSE QUESTIONS

Question 2

Tickets for the function are sold at the school office, the function hall and online.
Different prices are charged for students, teachers and parents.
Table 1 shows the number of tickets sold at each place and the total value of sales.

Table 1

	School office	**Function hall**	**Online**
Student tickets	283	35	84
Teacher tickets	28	4	3
Parent tickets	5	2	7
Total sales	\$8712	\$1143	\$2609

For this function

- student tickets cost \$$x$
- teacher tickets cost \$$y$
- parent tickets cost \$$z$.

a. Use the information in Table 1 to complete the following matrix equation by inserting the missing values in the shaded boxes.

$$\underbrace{\begin{bmatrix} 283 & 28 & 5 \\ 35 & 4 & 2 \\ 84 & 3 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 8712 \\ 1143 \\ 2609 \end{bmatrix}}_{k}$$

1 mark

b. Use the matrix equation to find the cost of a teacher ticket to the school function.

$X = A^{-1}K$. $\therefore X = \begin{bmatrix} 283 & 28 & 5 \\ 35 & 4 & 2 \\ 84 & 3 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 8712 \\ 1143 \\ 2609 \end{bmatrix} = \begin{bmatrix} 27 \\ 32 \\ 35 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\therefore y = 32$
cost of teacher ticket = \$32

Question 2

Rosa uses the following six-digit pin number for her bank account: 216342

With her knowledge of matrices, she decides to use matrix multiplication to disguise this pin number.

First she writes the six digits in the 2×3 matrix A.

$$A = \begin{bmatrix} 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

Next she creates a new matrix by forming the matrix product, $C = BA$,

where $B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$.

a. **i.** Determine the matrix $C = BA$.

$$C = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \end{bmatrix}$$

ii. From the matrix C, Rosa is able to write down a six-digit number that disguises her original pin number. She uses the same pattern that she used to create matrix A from the digits 216342.
Write down the new six-digit number that Rosa uses to disguise her pin number.

133926

1 + 1 = 2 marks

b. Show how the original matrix A can be regenerated from matrix C.

$$C = BA$$

$$\therefore B^{-1}C = B^{-1}.BA$$

$$\therefore B^{-1}C = I.A$$

$$\therefore A = B^{-1}C.$$

$$B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{1}{1\times-1-2\times-1} \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

Question 3

Market researchers claim that the ideal number of bookshops (x), sports shoe shops (y) and music stores (z) for a shopping centre can be determined by solving the equations

$$2x + y + z = 12$$
$$x - y + z = 1$$
$$2y - z = 6$$

a. Write the equations in matrix form using the following template.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \\ 6 \end{bmatrix}$$

$A. \quad X = K$

1 mark

b. Do the equations have a unique solution? Provide an explanation to justify your response.

det (A) = 1

Since det (A) ≠ 0 there will be

a unique solution.

1 mark

c. Write down an inverse matrix that can be used to solve these equations.

$$A^{-1} = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -2 & -1 \\ 2 & -4 & -3 \end{bmatrix}$$

1 mark

d. Solve the equations and hence write down the estimated ideal number of bookshops, sports shoe shops and music stores for a shopping centre.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -2 & -1 \\ 2 & -4 & -3 \end{bmatrix} \begin{bmatrix} 12 \\ 1 \\ 6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

$x = 3, y = 4, z = 2$

∴ 3 bookshops

4 sports shoe shops

2 music stores

Question 3

The basketball coach has written three linear equations which can be used to predict the number of points, p, rebounds, r, and assists, a, that Oscar will have in his next game.

The equations are

$$\begin{aligned} p + r + a &= 33 \\ 2p - r + 3a &= 40 \\ p + 2r + a &= 43 \end{aligned}$$

a. These equations can be written equivalently in matrix form.

Complete the missing information below.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} p \\ r \\ a \end{bmatrix} = \begin{bmatrix} 33 \\ 40 \\ 43 \end{bmatrix}$$

$A \cdot X = K$

1 mark

This matrix equation can be solved in the following way.

$$\begin{bmatrix} p \\ r \\ a \end{bmatrix} = \begin{bmatrix} 7 & -1 & -4 \\ -1 & 0 & 1 \\ x & 1 & 3 \end{bmatrix}\begin{bmatrix} 33 \\ 40 \\ 43 \end{bmatrix}$$

$X = A^{-1} K$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & -1 & 4 \\ -1 & 0 & 1 \\ -5 & 1 & 3 \end{bmatrix}$$

b. Determine the value of x shown in the matrix equation above.

$x = -5$

1 mark

c. How many rebounds is Oscar predicted to have in his next game?

$$\begin{bmatrix} p \\ r \\ a \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \\ 4 \end{bmatrix}$$

$\therefore r = 10$

He will have 10 rebounds

1 mark

The number of serves of bread (b), margarine (m), peanut butter (p) and honey (h) that contain, in total, 53 grams of fat, 101.5 grams of carbohydrate, 28.5 grams of protein and 3568 kilojoules of energy can be determined by solving the matrix equation

$A \cdot X = K$

$$\begin{bmatrix} 1.2 & 6.7 & 10.7 & 0 \\ 20.1 & 0.4 & 3.5 & 12.5 \\ 4.2 & 0.6 & 4.6 & 0.1 \\ 531 & 41 & 534 & 212 \end{bmatrix}\begin{bmatrix} b \\ m \\ p \\ h \end{bmatrix} = \begin{bmatrix} 53 \\ 101.5 \\ 28.5 \\ 3568 \end{bmatrix}$$

$X =$

$$\begin{bmatrix} b \\ m \\ p \\ h \end{bmatrix} = \begin{bmatrix} 1.2 & 6.7 & 10.7 & 0 \\ 20.1 & 0.4 & 3.5 & 12.5 \\ 4.2 & 0.6 & 4.6 & 0.1 \\ 531 & 41 & 534 & 212 \end{bmatrix}^{-1}\begin{bmatrix} 53 \\ 101.5 \\ 28.5 \\ 3568 \end{bmatrix}$$

$$\therefore \begin{bmatrix} b \\ m \\ p \\ h \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

Solve the matrix equation to find the values b, m, p and h.

$b = 4,\ m = 4,\ p = 2,\ h = 1$

2 marks