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ANSWERS

Question 9

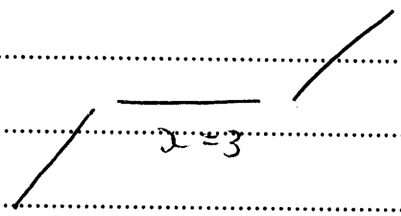
Consider the following derivative information for the function $f(x)$: (2 marks)

$$f'(x) > 0 \text{ for } x < 3$$

$$f'(x) = 0 \text{ at } x = 3$$

$$f'(x) > 0 \text{ for } x > 3$$

State the **nature** and the **location** of the stationary point.



At $x = 3$, there is a stationary point of inflection

Question 10

Differentiate $y = \frac{x^2 - x^3}{x} + \pi$; $x \neq 0$, with respect to x. (2 marks)

$$y = \frac{x^2 - x^3}{x} + \pi, \quad x \neq 0$$

$$\therefore y = x - x^2 + \pi, \quad x \neq 0$$

$$\therefore \frac{dy}{dx} = 1 - 2x, \quad x \neq 0$$

Section C continues opposite.

Question

Differentiate the function $f(x) = x^2 - 4x + 7$ from **First Principles**.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 4(x+h) + 7 - (x^2 - 4x + 7)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 4h}{h}$$

$$= \lim_{h \to 0} \frac{h(2x - 4 + h)}{h}$$

$$= \lim_{h \to 0} (2x - 4 + h)$$

$$= 2x - 4$$

$$\therefore f'(x) = 2x - 4.$$

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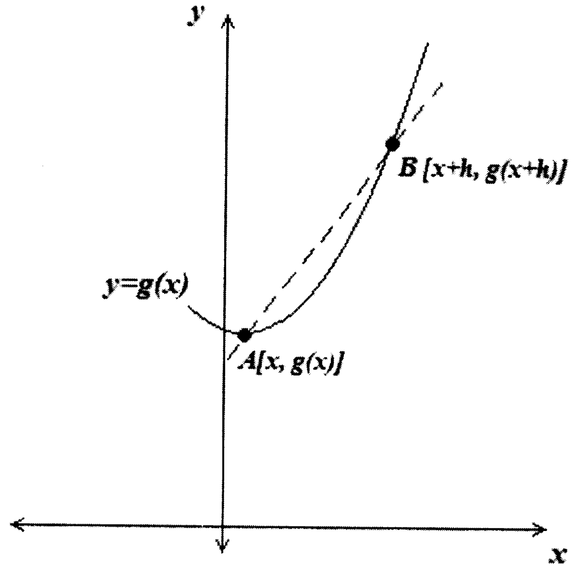
Section C (continued)

Question 12 (5 marks)



(a) Determine a fully simplified expression for the **gradient** of the **line segment through** ***AB*** on the curve $g(x) = 5 + \pi x^2$ as shown in the diagram above.

$$\frac{g(x+h) - g(x)}{h}$$

$$= \frac{5 + \pi(x+h)^2 - (5 + \pi x^2)}{h}$$

$$= \frac{\pi(x^2 + 2xh + h^2) - \pi x^2}{h}$$

$$= \frac{\pi(2xh + h^2)}{h}$$

$$= \frac{\pi h(2x + h)}{h}$$

$$= \pi(2x + h)$$

We are finding the gradient of the secant, so we do not have limit as h approaches zero.

Question 12 continues opposite.

Question 12 (continued)

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(b) What is the limiting condition that must be applied to determine an expression for $g'(x)$?

$$g'(x) = \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right)$$

(c) Apply this limiting condition to your simplified expression from (a), to determine by first principles an expression for $g'(x)$ and hence an exact value for $g'(3)$.

$$\lim_{h \to 0} \pi(2x + h)$$

$$\therefore \quad g'(x) = 2\pi x$$

$$g'(3) = 6\pi$$

No calculators

Question

Find the co-ordinates of the points on the curve with equation: $y = 2x^3 + 3x^2 - 12x + 7$ where the tangents are horizontal

$$\frac{dy}{dx} = 6x^2 + 6x - 12$$

For horizontal tangent, $\frac{dy}{dx} = 0$

$$\therefore \quad 6x^2 + 6x - 12 = 0$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2, 1$$

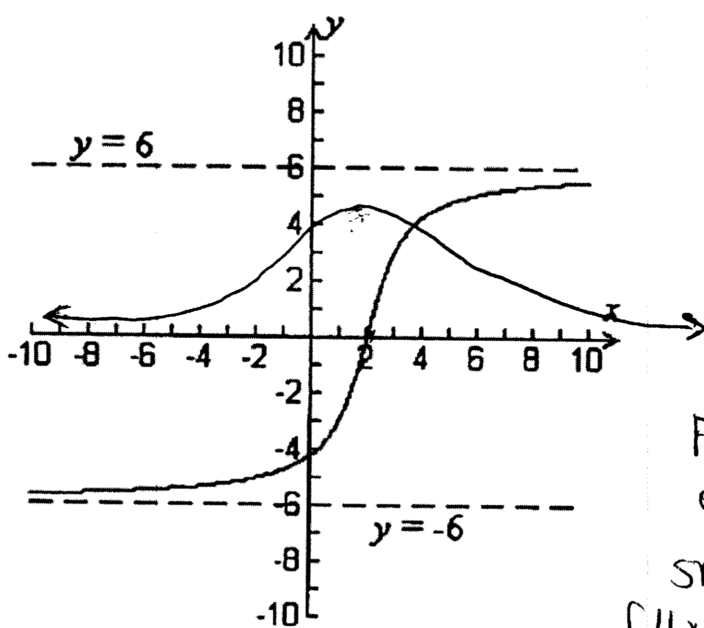
If $x = 1$, $y = 2 + 3 - 12 + 7 = 0$

If $x = -2$, $y = 2 \times -8 + 3 \times 4 + 12 \times 2 + 7$
$= -16 + 12 + 24 + 7$
$= 27$

$\therefore$ Points: $(1, 0)$ and $(-2, 27)$

Question

A particular function $g(x)$ has a graph as shown below. On the same axes, roughly sketch a graph of the derivative $g'(x)$. **(4 marks)**



$g'(x)$ is positive for all x.
For large values of x and very small values of x, $f'(x)$ approaches zero

f(x) is steepest at x = 2 (approximately) so
at x = 2 the derivaive has a maximum turning point.

No Calculators

Question

If $h'(x) = 4x^2 - 3x - 2$ and $h(-2) = 1$, then determine the function $h(x)$.

$$h(x) = \int 4x^2 - 3x - 2 \, dx$$

$$= \frac{4x^3}{3} - \frac{3x^2}{2} - 2x + c$$

$$h(-2) = 1 \quad \therefore \frac{4 \times -8}{3} - \frac{12}{2} + 4 + c = 1$$

$$\therefore -\frac{32}{3} - 6 + 4 + c = 1$$

$$\therefore c = 3 + \frac{32}{3} = \frac{41}{3}$$

$$\therefore h(x) = \frac{4x^3}{3} - \frac{3x^2}{2} - 2x + \frac{41}{3}$$

Question

The function $f(x) = 2x^2 + 3x + a$ has a tangent $y = 5 - x$. Determine the value of a.

$$f'(x) = 4x + 3$$

$\therefore$ When $f'(x) = -1$,

$$4x + 3 = -1$$

$$\therefore x = -1$$

$$f(-1) = 2(-1)^2 + 3(-1) + a$$

$$= 2 - 3 + a$$

$$= a - 1$$

$\therefore$ Point of contact is $(-1, a-1)$

But this point also lies on $y = 5 - x$

$$\therefore a - 1 = 5 - -1$$

$$a - 1 = 6 \quad \therefore a = 7$$

Section C (continued)

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Question 12 (4 marks)

The velocity $v(t)$, measured in kilometres per hour, of a dragster t seconds after starting a race is given by $v(t) = 0.8(200t - t^4)$. The acceleration of an object is given by $a(t) = v'(t)$.

(a) Determine an expression for the acceleration of the dragster.

$$\frac{dv}{dt} = 0.8(200 - 4t^3)$$

(b) Determine the acceleration of the dragster five seconds after the race begins and explain the significance of the sign of the acceleration. Include appropriate units.

$$a(5) = 0.8(200 - 4 \times 5^3)$$
$$= 0.8(200 - 4 \times 125)$$
$$= 0.8(200 - 500)$$
$$= 0.8 \times -300$$
$$= -240 \text{ km/hr/sec}$$

The acceleretion is to the left

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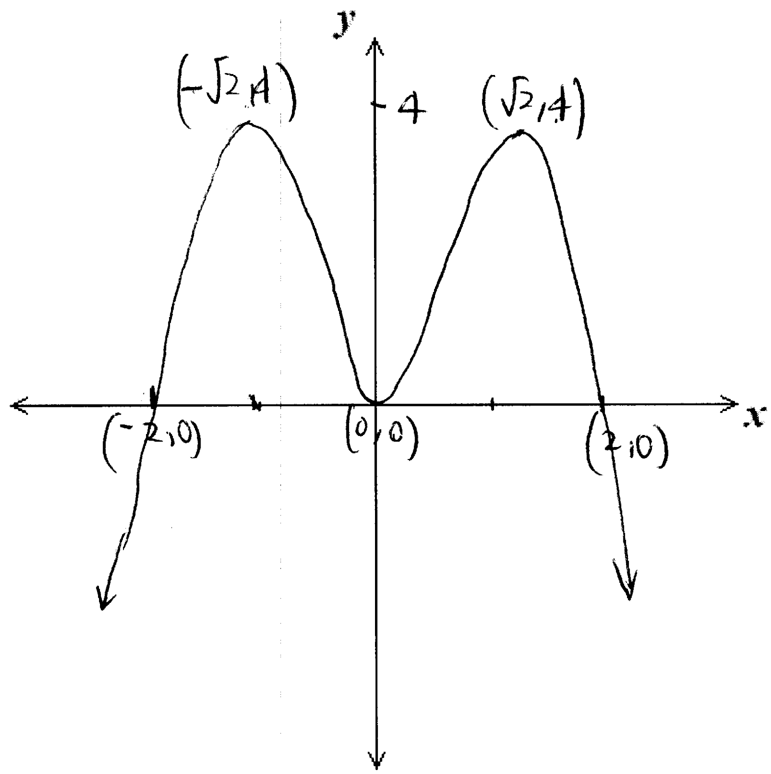
Section C (continued)

Question 11

Sketch the graph of $y = 4x^2 - x^4$ on the axes provided. Clearly label all intercepts and stationary points with exact values. (3 marks)



x-ints: $f(x)=0 \therefore 0 = 4x^2 - x^4$

$x^2(4-x^2)=0$

$x = 0, \pm 2$

$f'(x) = 8x - 4x^3$

$f'(x)=0$ if $4x(2-x^2)=0$

$x = 0, \pm\sqrt{2}$

$f(0)=0$

$f(\sqrt{2}) = 4\times(\sqrt{2})^2 - (\sqrt{2})^4$

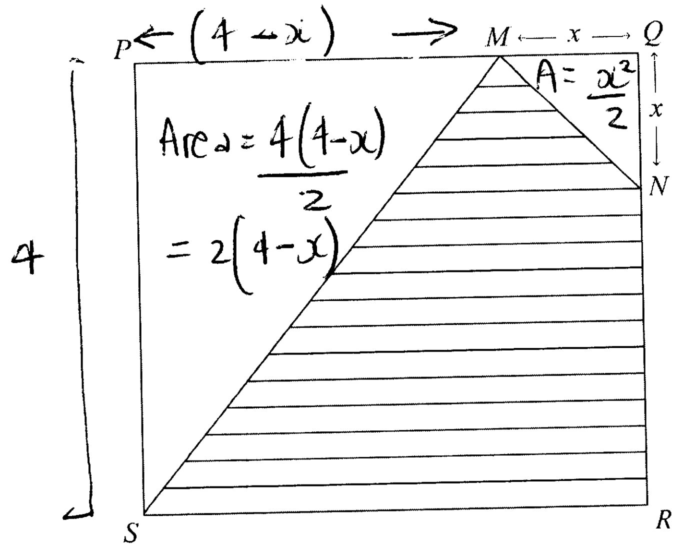
$= 4\times 2 - 4$

$= 4$

$\therefore (+\sqrt{2},4)$ and $(-\sqrt{2},4)$ are stationary points

Question 11

PQRS is a square of side length 4 metres. The lengths *MQ* and *QN* are each x metres.



a. Write down an expression for the length *PM* in terms of x.

$4 - x$

1 mark

b. Show that the shaded area, A, is given by $A = 8 + 2x - \frac{1}{2}x^2$.

Shaded area = A square − A triangle 1 − A triangle 2

$= 16 - \frac{x^2}{2} - 2(4-x)$

$= 16 - \frac{x^2}{2} - 8 + 2x$

$\therefore A = 8 + 2x - \frac{x^2}{2}$

1 mark

c. Find the maximum value of A.

$A'(x) = 2 - x$

$= 0$ if $x = 2$

$A(2) = 8 + 2 \times 2 - \frac{2^2}{2}$

$= 8 + 4 - 2$

$= 10$ sq. units

2 marks

Total 4 marks

END OF QUESTION AND ANSWER BOOKLET

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10. For the function

$f:[-\pi,\pi]\to R, f(x)=-3\cos(2x)$

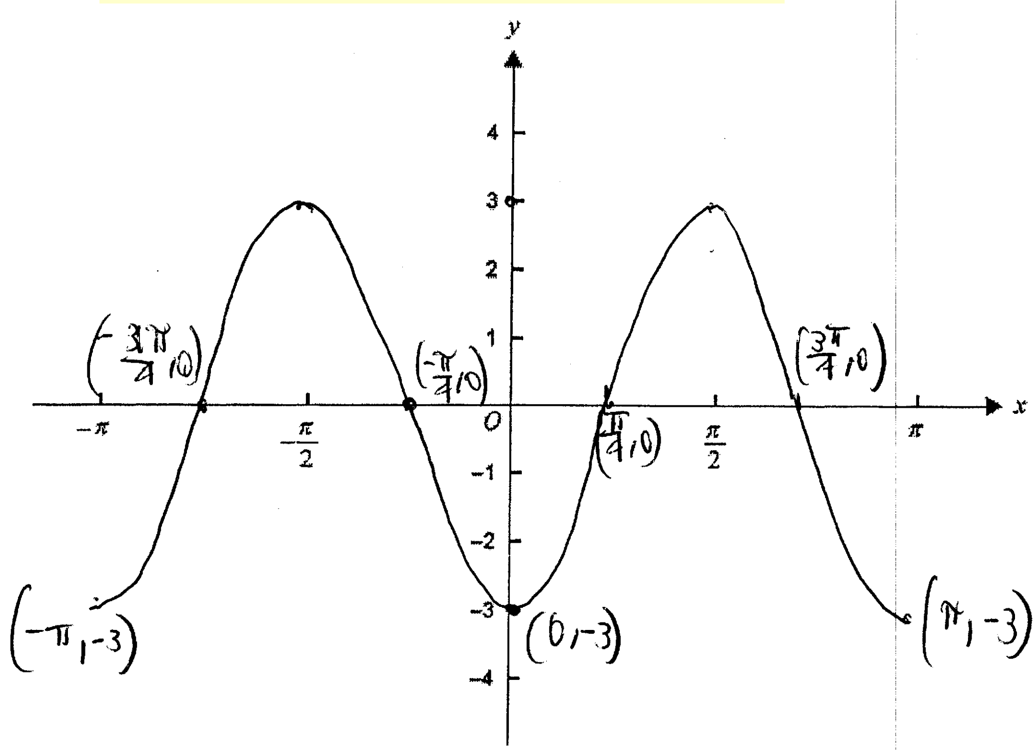
a. write down the period and the amplitude of the function

Period $= \frac{2\pi}{2} = \pi$

Amplitude $= 3$

2 marks

b. sketch the graph of the function f on the set of axes below. Label the intercepts and any endpoints with their co-ordinates.



2 marks

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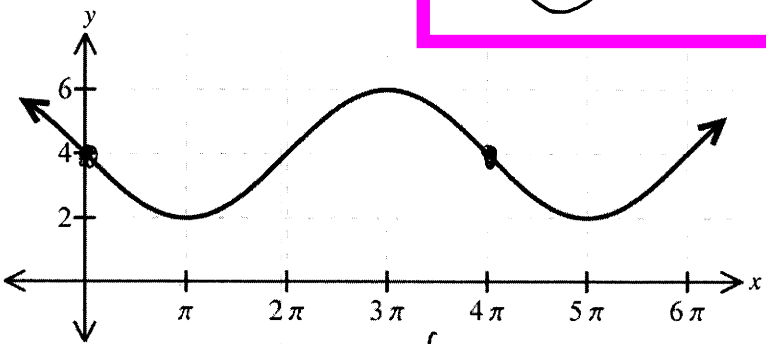
Section B (continued)

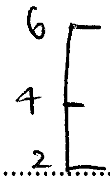
Question 10 (3 marks)

Negative sine

The graph below is of the form $y = a\sin(bx) + c$.



Find the exact values of a, b and c.



$a = -2$

Period $= 4\pi$

$\therefore \dfrac{2\pi}{b} = 4\pi$

$\therefore b = \dfrac{1}{2}$

$c = 4$

$a = -2,\ b = \dfrac{1}{2},\ c = 4$

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Section B

Answer **ALL** questions in this section.

This section assesses **Criterion 4**.

Question 5

(2 marks)

(a) Convert $\frac{3\pi}{5}$ radians into degrees.

$\frac{3\pi}{5} \times \frac{180}{\pi} = \frac{3}{5} \times 180 = 108^\circ$

(b) Convert 18° into radians.

$18^\circ \times \frac{\pi}{180} = \frac{\pi}{10}^c$

Question 6

(2 marks)

Evaluate $\tan\frac{5\pi}{6} + \cos\frac{-\pi}{4}$.

Express answer as an exact value.

$\tan\left(\frac{5\pi}{6}\right) + \cos\left(\frac{-\pi}{4}\right)$

$= \frac{-1}{\sqrt{3}} + \frac{\sqrt{2}}{2}$

Section B continues opposite.

Section B (continued)

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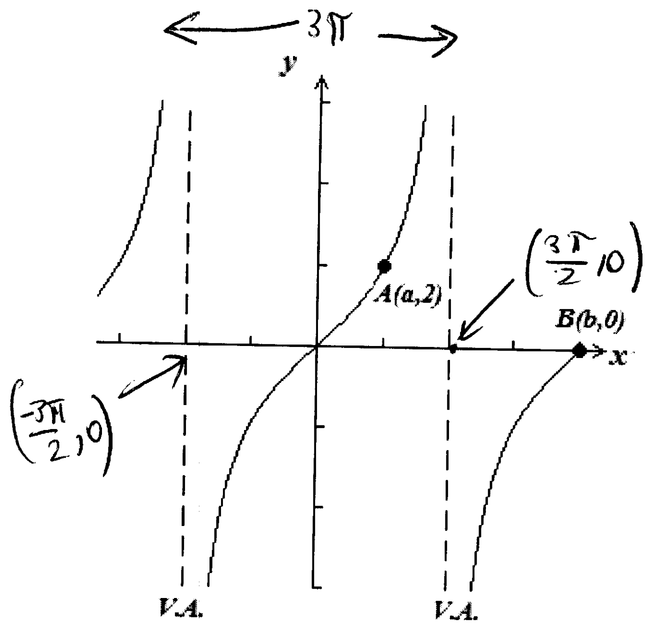
Question 7 (3 marks)



The sketch above is of the function $y = 2\tan\left(\frac{x}{3}\right)$. $\rightarrow$ Period $= \frac{\pi}{1/3} = 3\pi$

Point A has co-ordinates $(a, 2)$.

Point B has co-ordinates $(b, 0)$.

At A, $\tan\left(\frac{x}{3}\right) = 1$

$\therefore \frac{x}{3} = \frac{\pi}{4} \quad \therefore x = \frac{3\pi}{4}$

(a) Find the exact values of a and b.

$a = \frac{3\pi}{4}$

$b = \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi$

(b) Determine the exact values for the asymptotes shown.

$x = \frac{3\pi}{2}, \quad x = \frac{-3\pi}{2}$

Section B continues over the page.

For Marker Use Only

Section B

Answer **ALL** questions in this section.

This section assesses **Criterion 5**.

Question 5 (4 marks)

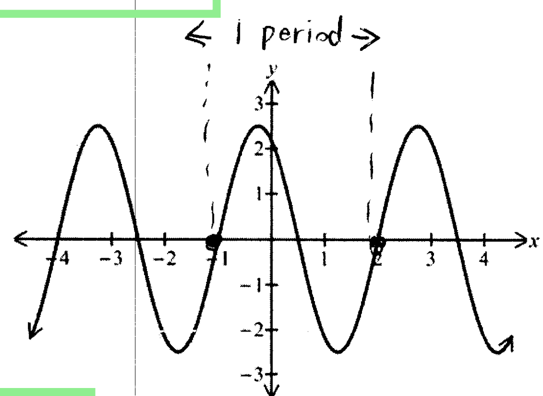
State the exact value of the period and amplitude of:

(a) the function $f(x) = 4 - \sin(6x)$

Period $\frac{2\pi}{6} = \frac{\pi}{3}$

Amplitude 1

(b) the graph



Period 3

Amplitude 2.5

Question 6 (2 marks)

If $\cos\theta = 0.3$, then determine $1 - \cos^2\left(\frac{\pi}{2} - \theta\right)$.

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\therefore \quad 1 - \cos^2\left(\frac{\pi}{2} - \theta\right) = 1 - \sin^2\theta$

$= \cos^2\theta$

$= (0.3)^2 = 0.09$

Section B continues.

Section B (continued)

Question 8

(5 marks)

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Given $\sin\left(\frac{5\pi}{4}\right)+\tan(2\theta)+\cos\left(-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{6}-\frac{\sqrt{2}}{2}$,

determine **exact values** for θ, if $-\pi \le \theta \le \pi$.

$\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\cos\left(\frac{-\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\therefore -\frac{\sqrt{2}}{2} + \tan(2\theta) + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{6} - \frac{\sqrt{2}}{2}$

$\therefore \tan(2\theta) = \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{2}$

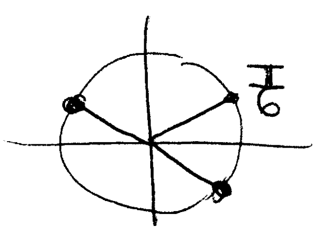
$\tan(2\theta) = \frac{\sqrt{3}}{6} - \frac{3\sqrt{3}}{6}$

$\tan(2\theta) = -\frac{2\sqrt{3}}{6}$

$\tan(2\theta) = -\frac{\sqrt{3}}{3}$, $-\pi \le \theta \le \pi$

$\therefore -2\pi \le 2\theta \le 2\pi$

Remember to adjust domain



$2\theta = -\frac{\pi}{6}, -\frac{7\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$

$\therefore \theta = \frac{-\pi}{12}, \frac{-7\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}$

Question

Water is being collected in a tank. The volume V cubic metres of water in the tank after t minutes is given by: $V = 2t^2 - 3t + 1$. Find:

a. The average rate of change of V between $t = 1$ and $t = 3$.

$$\frac{V(3)-V(1)}{3-1} = \frac{10-0}{2} = 5 \ m^3/min$$

$$V(3) = 2\times3^2 - 3\times3+1$$
$$= 18-9+1 = 10$$
$$V(1) = 2\times1^2 - 3\times1+1$$
$$= 2-3+1$$
$$= 0$$

b. The instantaneous rate of change of V at $t = 3$.

$$\frac{dV}{dt} = 4t - 3$$

$$\left.\frac{dV}{dt}\right|_{t=3} = 4\times3-3 = 9 m^3/min$$

Question

The line with equation $y = 4x - 5$ is tangent to the curve $y = x^4 + c$. Find the value of c.

$$y = x^4 + c \quad \frac{dy}{dx} = 4x^3 \qquad \text{When} \quad 4x^3 = 4$$
$$x = 1$$

On the line $y = 4x - 5$, when $x = 1$,
$$y = 4\times1-5 = -1$$

$\therefore (1,-1)$ lies on $y = x^4 + c$

$$\therefore \quad -1 = 1^4 + c$$
$$\therefore \quad c = -2$$

Question

If $f(x) = -\frac{3}{2}\sin(2x) + \frac{3}{4}$, find the smallest value of x so that $f(x) = 0$ for $x \in [0,2\pi]$

$$-\frac{3}{2}\sin(2x) + \frac{3}{4} = 0, \qquad 0 \le x \le 2\pi$$

$$\therefore \quad \sin(2x) = \frac{1}{2}, \qquad 0 \le 2x \le 4\pi$$

$$\therefore \quad 2x = \frac{\pi}{6} \quad \text{is the first value}$$

$$x = \frac{\pi}{12}$$

Question 10

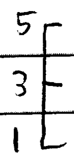
a. If $f:[0,\pi] \to R$, $f(x) = 3 - 2\cos\left(\frac{x}{2}\right)$.

i. State the amplitude and period of f.

$a = 2$

$\text{Period} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

ii. Determine the range of f.

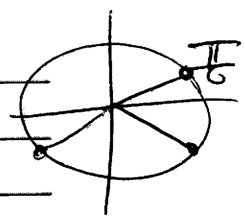


$\text{Range}: [1, 5]$

2 + 2 = 4 marks

b. Solve the equation $2\sin(x) = -1$ for $x \in [-\pi, \pi]$.

$\sin(x) = -\frac{1}{2}$



$x = -\frac{\pi}{6}, -\frac{5\pi}{6}$

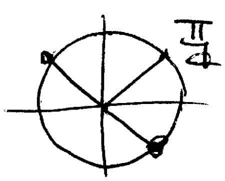
3 marks

Question

Solve the equation:

$\tan(3x) = -1$ for $-\pi < x < \pi$

$\tan(3x) = -1 \quad , -3\pi < x < 3\pi$



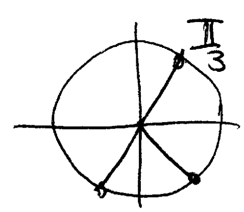
$3x = -\frac{\pi}{4}, -\frac{5\pi}{4}, -\frac{9\pi}{4},$
$\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{\cancel{9}\pi}{4}, \frac{11\pi}{4}$

$\boxed{x = -\frac{\pi}{12}, -\frac{5\pi}{12}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}}$

Question

Solve the equation: $2\sin(2x) + \sqrt{3} = 0$ for $0 < x < \pi$

$\sin(2x) = -\frac{\sqrt{3}}{2} \quad 0 < 2x < 2\pi$



$2x = \frac{4\pi}{3}, \frac{5\pi}{3}$

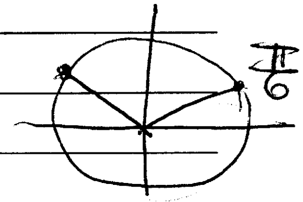
$\boxed{\therefore x = \frac{2\pi}{3}, \frac{5\pi}{6}}$

Question 9

a. Solve the equation $3\sin\left(\frac{x}{2}\right) - \frac{3}{2} = 0$ for $x \in [-\pi, 2\pi]$.

$\sin\left(\frac{x}{2}\right) = \frac{1}{2}$, $-\frac{\pi}{2} \leq \frac{x}{2} \leq \pi$

$\frac{x}{2} = \frac{\pi}{6}, \frac{5\pi}{6}$



$x = \frac{\pi}{3}, \frac{5\pi}{3}$

2 marks

b. What is the average rate of change of $f(x) = -3\sin\left(\frac{x}{2}\right)$ between $x = -\pi$ and $x = \pi$?

$\frac{f(\pi) - f(-\pi)}{\pi - -\pi}$

$f(\pi) = -3\sin\left(\frac{\pi}{2}\right)$
$= -3$

$= \frac{-3-3}{2\pi} = \frac{-6}{2\pi}$

$f(-\pi) = -3\sin\left(\frac{-\pi}{2}\right)$
$= 3$

$= \frac{-3}{\pi}$

2 marks

c. If $\cos(x) = 0.7$, where $0 < x < \frac{\pi}{2}$, evaluate $\cos(2\pi - x) - \cos(\pi + x)$.

$\cos(2\pi - x) - \cos(\pi + x)$

$= \cos(x) - -\cos(x)$

$= 2\cos(x)$

$= 1.4$

2 marks
Total 6 marks

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7. Given that $\sin\theta = -\frac{5}{13}$ and $\frac{3}{2}\pi < \theta < 2\pi$, find the value of $\cos\theta$

$\rightarrow$ 4th Quadrant, $\cos\theta > 0$

$$\cos^2\theta = 1 - \left(\frac{-5}{13}\right)^2$$
$$= 1 - \frac{25}{169}$$
$$= \frac{144}{169}$$

$\therefore \cos\theta = \sqrt{\frac{144}{169}}$ (4th Quadrant)

so cosine is positive
Take positive
square root only

$\therefore \cos\theta = \frac{12}{13}$

3 marks

8. If $f(x) = \frac{\sqrt{x}}{3}\left(x\sqrt{x} + 2x^{\frac{5}{2}}\right)$ find the value of $f'(-3)$

$$f(x) = \frac{x^{1/2}}{3}\left(x \cdot x^{1/2} + 2x^{5/2}\right)$$
$$= \frac{1}{3}\left(x \cdot x^{1/2+1/2} + 2x^{5/2+1/2}\right)$$
$$= \frac{1}{3}\left(x^2 + 2x^3\right)$$
$$\therefore f'(x) = \frac{1}{3}\left(2x + 6x^2\right)$$
$$\therefore f'(-3) = \frac{1}{3}\left(2\times -3 + 6\times 9\right)$$
$$= \frac{1}{3}(-6+54) = 16$$

3 marks

9. (i) Find $\int (2-x)^2 \, dx$

$$\int (2-x)^2 dx = \int 4 - 4x + x^2 \, dx$$
$$= 4x - 2x^2 + \frac{x^3}{3} + c$$

(ii) If $f'(x) = (2-x)^2$, find $f(x)$ given that $f(3) = 0$

$$f(x) = 4x - 2x^2 + \frac{x^3}{3} + c$$
$$f(3) = 0 \quad \therefore 0 = 4\times 3 - 2\times 9 + \frac{27}{3} + c$$
$$0 = 12 - 18 + 9 + c$$
$$\therefore 0 = 3 + c \quad \therefore c = -3$$

3 marks

$$\therefore f(x) = 4x - 2x^2 + \frac{x^3}{3} - 3$$