Chapter 1 Review

SOLUTIONS

Q10.

$$f(x) = \frac{ax+b}{cx+d}.$$

$$= \frac{ax + \frac{da}{c}}{cx+d} + \frac{b - \frac{da}{c}}{cx+d}$$

$$= \frac{a\left(x + \frac{d}{c}\right)}{cx+d} + \frac{b - \frac{da}{c}}{cx+d}$$

$$= \frac{\frac{a}{c}(cx+d)}{cx+d} + \frac{b - \frac{da}{c}}{cx+d}$$

$$= \frac{a}{c} + \frac{b - \frac{da}{c}}{cx + d}$$

In this format, it is much easier to identify the domain and range.

DOMAIN RANGE TABLE

dom(f)	ran(f)
$\mathbb{R} \setminus \{-\frac{d}{c}\}$	$\mathbb{R} \setminus \{\frac{a}{c}\}$
dom(f^{-1})	ran(f^{-1})
$\mathbb{R} \setminus \{\frac{a}{c}\}$	$\mathbb{R} \setminus \{-\frac{d}{c}\}$

$$y = \frac{ax+b}{cx+d}$$

Interchange x and y:

$$x = \frac{ay+b}{cy+d}$$

$$(cy+d)x = ay+b$$

$$cyx + dx = ay+b$$

$$cxy - ay = b - dx$$

$$y(cx-a) = b - dx$$

$$y = \frac{b-dx}{cx-a}$$

$$f^{-1}(x) = \frac{b-dx}{cx-a}$$

Our fully specified inverse function is:

$$f^{-1}: \mathbb{R} \setminus \left\{\frac{a}{c}\right\} \longrightarrow \mathbb{R},$$
$$f^{-1}(x) = \frac{b - dx}{cx - a}$$

(b)(i) If $a = 3$, $b = 2$, $c = 3$, $d = 1$

We substitute these values into the expression obtained above for f^{-1}:

$$f^{-1}(x) = \frac{b - dx}{cx - a}, \quad x \neq \frac{a}{c}$$

So, $$f^{-1}(x) = \frac{2 - 1.x}{3x - 3}, \quad x \neq \frac{3}{3}$$

$$f^{-1}(x) = \frac{2 - x}{3(x - 1)}, \quad x \in \mathbb{R} \setminus \{1\}$$

(b) (ii) Same method as for b (i):

If $a = 3, b = 2, c = 2, d = -3,$

$$f^{-1}(x) = \frac{b - dx}{cx - a}$$

$$= \frac{2 + 3x}{2x - 3},$$

$$x \in \mathbb{R} \setminus \left\{ \frac{3}{2} \right\}$$

$$\boxed{\text{(b) (iii) and (iv) are done in exactly the same way!!}}$$

(c) We are given that $f(x) \equiv f^{-1}(x)$ (that is, $f(x)$ and $f^{-1}(x)$ are the same function)

We have established:

$$f(x) = \frac{ax+b}{cx+d}$$

$$f^{-1}(x) = \frac{dx-b}{a-cx}$$

If these two functions are to be the same, then:

$$\frac{ax+b}{cx+d} = \frac{dx-b}{a-cx}$$

for all values of x

Cross multiply:

$$(ax+b)(a-cx)=(dx-b)(cx+d)$$

$$a^2x-acx^2+ba-bcx$$
$$=dcx^2+d^2x-bcx-bd$$

Now simplify by collecting like terms and make the RHS zero:

$$a^2x-d^2x-acx^2-dcx^2+ba+bd=0$$

$$x(a^2-d^2)-x^2(ac+dc)+(ba+bd)=0$$

For this above statement to hold true for ALL values of x, the coefficients of x and x^2, as well as the constant term, must all equal zero

This gives us three equations which must all be satisfied:

$ac + dc = 0$ (coefficient of x^2)

$d^2 - a^2 = 0$ (coefficient of x)

$ba + bd = 0$ (constant term)

$ac + dc = 0 \quad \therefore c(a+d) = 0$

$c = 0$ or

$d = -a$

$d^2 - a^2 = 0 \quad \therefore (d-a)(d+a) = 0$

$d = a$ or $d = -a$

$ba + bd = 0 \quad \therefore b(a+d) = 0$

$b = 0$ or $a = -d$

The condition common to all three is: $a = -d$

Thus, if $a = -d$ then $f(x)$ and $f^{-1}(x)$ will be the same function.