Section A: Multiple Choice

Question 1

The average rate of change of the function $f(x) = 2x^3 - 4x^2$ on the interval [1,3] is:

A. $6x^2 - 8x$
B. 46
C. 20
D. 10
E. 9

$$\frac{f(3) - f(1)}{3 - 1} = \frac{(2 \times 3^3 - 4 \times 3^2) - (2 \times 1^3 - 4 \times 1^2)}{2}$$

$$= \frac{(54 - 36) - (2 - 4)}{2}$$

$$= \frac{20}{2} = 10$$

Question 2

The equation of the normal to the curve $y = 4x - 3x^2$ at the point (1,1) is:

A. $y = \frac{1}{2}x + \frac{1}{2}$
B. $y = -2x$
C. $y = 2x + 3$
D. $y = -2x + 3$
E. $y = -\frac{1}{2}x + 1$

$$\frac{dy}{dx} = 4 - 6x$$

$$\left.\frac{dy}{dx}\right|_{x=1} = -2$$

$$\therefore m_{NORMAL} = \frac{-1}{-2} = \frac{1}{2}$$

$\therefore$ A is correct option

Question 3

The equation of the tangent to the curve: $y = -2x + 4x^2$ at the point (-1,6) is

A. $y = (-2 + 8x)x - 4$
B. $y = -8x - 2$
C. $y = -6x$
D. $y = 6x + 12$
E. $y = -10x - 4$

$$\frac{dy}{dx} = -2 + 8x$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = -2 - 8 = -10$$

$\therefore$ E is correct as it is the only linear equation with $m = -10$

Question 4

The function with equation: $y = x^3 - 48x$ has stationary points at:

A. $x = 0, \sqrt{48}, -\sqrt{48}$
B. 0 and 4
C. 0 and – 4
D. 4 and – 4
E. 4 only

$$\frac{dy}{dx} = 3x^2 - 48$$

For stationary points, $\frac{dy}{dx} = 0$

$$\therefore 3x^2 - 48 = 0$$

$$x^2 = 16$$

$x = \pm 4$

Question 5

If $\frac{dy}{dx} = 6x^2 - 1$ and $y = 0$ when $x = 2$ then an expression for y is:

A. $y = 12x$
B. $y = 12x - 24$
C. $y = 2x^3 - x$
D. $y = 2x^3 - x - 14$
E. $y = 2x^3 + x - 18$

$y = \int 6x^2 - 1 \, dx$
$= 2x^3 - x + C$
When $x = 2$, $y = 0$
$\therefore 0 = 2 \times 2^3 - 2 + C$
$0 = 14 + C$
$\therefore C = -14$

Question 6

The volume V litres of water in a tank over a period of t months is given by the formula: $V(t) = t^3 - t^2 + 2t + 4, 0 \le t < 4$. The rate at which the tank is filling at $t = 2$ is:

A. 60 l/month
B. 30 l/month
C. 10 l/month
D. 12 l/month
E. 0 l/month

$\frac{dV}{dt} = 3t^2 - 2t + 2$

Since they are asking for the rate at a single instant in time, it is the instantaneous rate of change

$V'(2) = 3 \times 2^2 - 2 \times 2 + 2$
$= 12 - 4 + 2$
$= 10$

Question 7

The maximum value of the function $y = 10x - 4x^2$ is:

A. 1.25
B. 6.25
C. 2.5
D. 0
E. 10

The maximum value of a function is the y-value that has the greatest value on its domain.

$f'(x) = 10 - 8x$
$f'(x) = 0$ when $x = \frac{5}{4}$
$f\left(\frac{5}{4}\right) = \frac{10 \times 5}{4} - 4 \times \left(\frac{5}{4}\right)^2 = \frac{50}{4} - \frac{25}{4}$
$= \frac{25}{4}$

Question 8

The tangent to the curve $y = 2x(x^2 - 5)$ at point P is parallel to the line with equation $y = -4x$. The co-ordinates of the point P could be:

A. $(-1,8)$
B. $(-4,-72)$
C. $(2,-4)$
D. $(0,0)$
E. $(3,24)$

$y = 2x^3 - 10x$
$\frac{dy}{dx} = 6x^2 - 10$
$6x^2 - 10 = -4$
$\therefore 6x^2 = 6$
$\therefore x^2 = 1$
$x = \pm 1$
$\therefore$ Only A has the necessary x-value.

Section B: Short Answer Questions

Question 1

Let , $g(x) = x^3 - 5x^2 + 7x - 3$.

a. Use calculus to determine the co-ordinates of the stationary points of this function.

$g'(x) = 3x^2 - 10x + 7$

Let $g'(x) = 0$

$\therefore 3x^2 - 10x + 7 = 0$

$(3x - 7)(x - 1) = 0$

$x = \frac{7}{3}, 1$

$g\left(\frac{7}{3}\right) = -\frac{32}{27}$

$g(1) = 0$

$\therefore$ Stationary points:

$\left(\frac{7}{3}, \frac{-32}{27}\right), (1, 0)$

4 marks

b. State the co-ordinates of the y-intercept

$(0, -3)$

1 mark

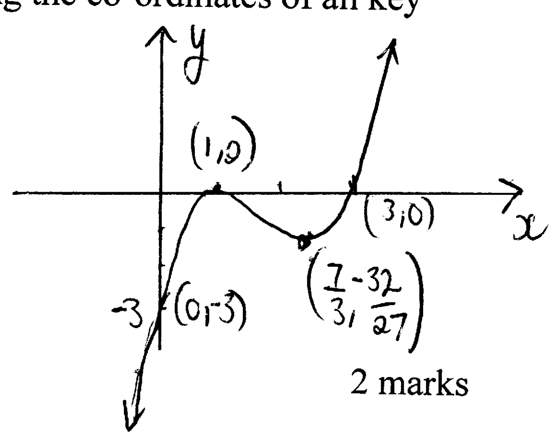
c. State the co-ordinates of the x-intercepts.

$g(x) = 0 \quad x = 1, 3$

$\therefore (1,0), (3,0)$

1 mark

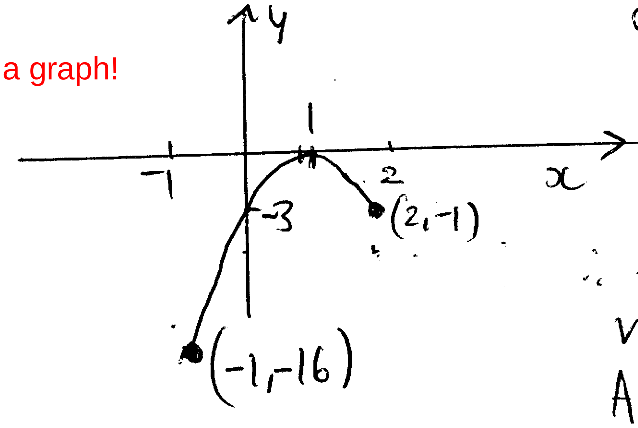
d. Sketch the graph of this function, indicating the co-ordinates of all key points on your sketch.



2 marks

e. The domain of the function $g(x)$ above is now restricted so that $x \in [-1,2]$. Determine the absolute maximum and the absolute minimum of the resulting function.

Sketch a graph!



$g(-1) = -16$

$g(2) = -1$

$g(1) = 0$

$\therefore$ Absolute maximum value $= 0$

Absolute minimum $= -16$

3 marks

Question 2

The function with equation: $h(x) = x^3 + ax^2 + bx + 1$ has a stationary point at (2,–11). Find the values of a and b.

$h'(2) = 0$ $\qquad$ $h'(x) = 3x^2 + 2ax + b$

$h(2) = -11$

$\therefore 0 = 12 + 4a + b$

$\therefore -12 = 4a + b$ ①

$-11 = 2^3 + 4a + 2b + 1$

$\therefore -11 = 9 + 4a + 2b$

$\therefore -20 = 4a + 2b$

$\therefore -10 = 2a + b$ ②

4 marks

① − ②: $-2 = 2a$

$$\boxed{\begin{aligned} &\therefore a = -1 \\ &\therefore b = -8 \end{aligned}}$$

Question 3

Consider the function with equation: $f(x) = (x-a)^3(x-b)$ where a and b are positive constants with $b > a$

a. Find the derivative function $f'(x)$

$$f'(x) = (x-a)^2(4x-a-3b)$$

Define f(x) and use CAS to differentiate

1 mark

b. Find the co-ordinates of the stationary points of f in terms of a and b.

Let $f'(x)=0 \quad \therefore (x-a)^2(4x-a-3b)=0$

$x = a, \ \frac{a+3b}{4}$

$f(a)=0 \quad f\left(\frac{a+3b}{4}\right) = \frac{-27(a-b)^4}{256}$

2 marks

c. Find the values of a and b if the stationary points occur at $x = 3$ and $x = 4$.

Stationary points: $(a, 0), \left(\frac{a+3b}{4}, \frac{-27(a-b)^4}{256}\right)$

(c) Since $b > a$, we must have $a = 3$.

2 marks

$\therefore 4 = \frac{3+3b}{4}$

$16 = 3 + 3b$

$\therefore b = \frac{13}{3}$

$\therefore a = 3, \ b = \frac{13}{3}$

Section D Analysis Section

Question 1

This tells us x = 0 when t=0

A particle moves in a straight line and starts from a fixed point O on the line. Its acceleration a m/s^2 is given by: $a = 24t - 72$ for any time t seconds where $t \geq 0$.

a. What is its acceleration at time $t = 5$?

$a(5) = 24 \times 5 - 72$

$= 120 - 72 = 48 \text{ m/s}^2$

1 mark

b. If its initial velocity is 96m/s, what is its velocity for any time t seconds?

$V = \int 24t - 72 \, dt$

$= 12t^2 - 72t + C$

When $t = 0$, $V = 96$ $\therefore$ $C = 96$

$\therefore V = 12t^2 - 72t + 96$

3 marks

c. At what values of t is the particle instantaneously at rest?

Let $V = 0$ $\therefore$ $12t^2 - 72t + 96 = 0$

$12(t^2 - 6t + 8) = 0 \therefore 12(t-4)(t-2) = 0$

$\therefore t = 2, 4$

2 marks

d. What is its displacement from O at any time t?

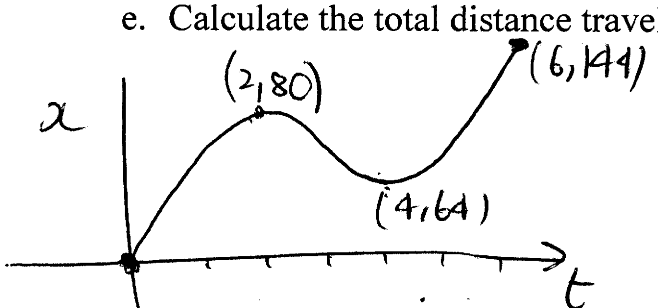
$x = \int 12t^2 - 72t + 96 \, dt$

$= 4t^3 - 36t^2 + 96t + C$

At $t = 0$, $x = 0$ $\therefore C = 0$

$\therefore x = 4t^3 - 36t^2 + 96t$

e. Calculate the total distance travelled by the particle in the first 6 seconds.



$x = 4t^3 - 36t^2 + 96t$

$x(2) = 80$

$x(4) = 64$

$x(6) = 144$

Distance

$= 80 + 16 + (144 - 64)$

$= 176$ m

3 marks