CHAPTER 11: Integration (Antidifferentiation)

Antidifferentiation is the inverse operation to differentiation.

Integration means exactly the same thing as antidifferentiation

Differentiate command: $\frac{d}{dx}(f(x))$	Antidifferentiate command: $\int f(x)dx$
$\frac{d}{dx}\big(f(x) \pm g(x)\big) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$	$\int \big(f(x) \pm g(x)\big)dx = \int f(x)dx \pm \int g(x)dx$
$\frac{d}{dx}\big(kf(x)\big) = kf'(x)$, where *k* is a constant.	$\int kf(x)dx = k\int f(x)dx$, where *k* is a constant.

INDEFINITE INTEGRAL (ANTIDERIVATIVE)	**DEFINITE INTEGRAL**
The indefinite integral (or the antiderivative) is always a **function**. A constant, $c \in R$, is also introduced by the antidifferentiation process. The antiderivative of f is written as: $\int f(x)dx$	The definite integral always gives a **numerical value** that can be interpreted as a signed area between a curve and the x-axis. It is written: $\int_a^b f(x)dx$ The numbers *b* and *a* are called terminals.

THE INDEFINITE INTEGRAL

INDEFINITE INTEGRAL questions can be asked in several different ways:

Question 1

Find **the** antiderivative of $g(x) = 6x^2 - 4x + 8$

Question 2

Find **an** antiderivative of $g(x) = 6x^2 - 4x + 8$

Question 3

Determine the function $g(x)$ given that: $g'(x) = 6x^2 - 4x + 8$ and $g(1) = 3$.

Here is the Calculus section of the VCAA Formula Sheet that you will receive in BOTH exams:

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an\left(ax+b\right)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}(\sin(ax)) = a\ \cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}(\cos(ax)) = -a\ \sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$		

Question 2

Find an anti-derivative of $\frac{1}{(2x-1)^3}$ with respect to x.

2 marks

Question 2 (3 marks)

Let $f'(x) = 1 - \frac{3}{x}$, where $x \neq 0$.

Given that $f(e) = -2$, find $f(x)$.

NOTE: $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b)$

But this does NOT appear on the VCAA formula sheet

Question 2

a. Find an antiderivative of $\frac{6}{3x-4}$ with respect to x.

1 mark

Question 7 (3 marks)

If $f'(x)=2\cos(x)-\sin(2x)$ and $f\left(\frac{\pi}{2}\right)=\frac{1}{2}$, find $f(x)$.

ANTIDIFFERENTIATION BY RECOGNITION

There are frequently exam questions which ask you to differentiate a function and then use your result to work out the antiderivative of a related function. These questions can be easily recognized from the layout:

1) They ask you to differentiate something
2) They then say : "….**hence**, find the antiderivative of…."

Example:

a. Differentiate the function: $x\cos(x)$

b. **Hence**, find an antiderivative of $x\sin(x)$

THE DEFINITE INTEGRAL

- The definite integral always gives as its output a numerical value.
- There is **NO** constant of integration involved.
- The value of the definite integral can be interpreted as the signed area between a function and the x-axis over a domain [*a*, *b*]. This area is positive if *f*(*x*) is above the x-axis and negative if it is below the x-axis.

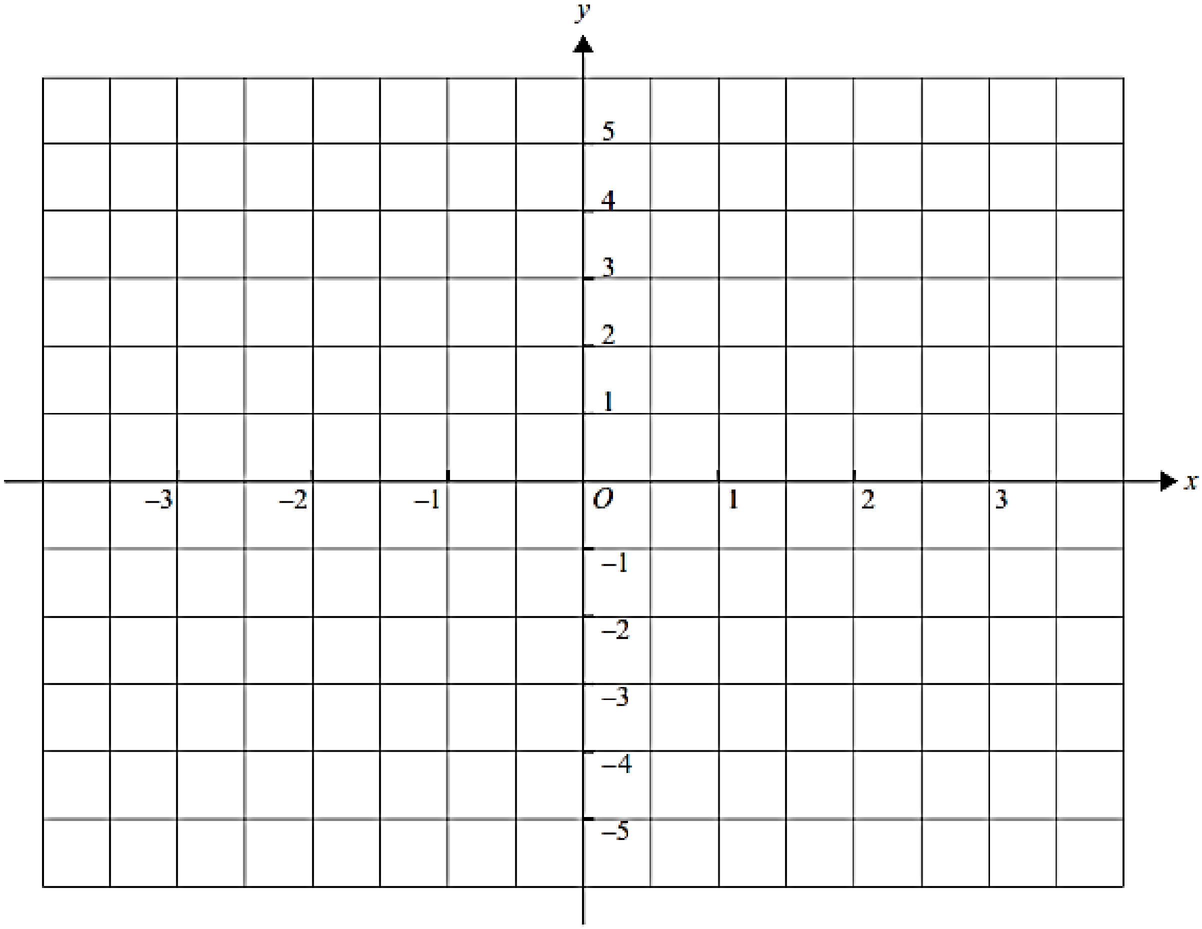
$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$ where $F(x)$ is the antiderivative of $f(x)$.

Note: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Question 3 (5 marks)

Let $f: R\backslash\{1\} \rightarrow R$, where $f(x) = 2 + \dfrac{3}{x-1}$.

a. Sketch the graph of f. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation. 3 marks



b. Find the area enclosed by the graph of f, the lines $x = 2$ and $x = 4$, and the x-axis. 2 marks

Question 2 (2 marks)

Let $\int_4^5 \frac{2}{2x-1}\,dx = \log_e(b)$.

Find the value of b.

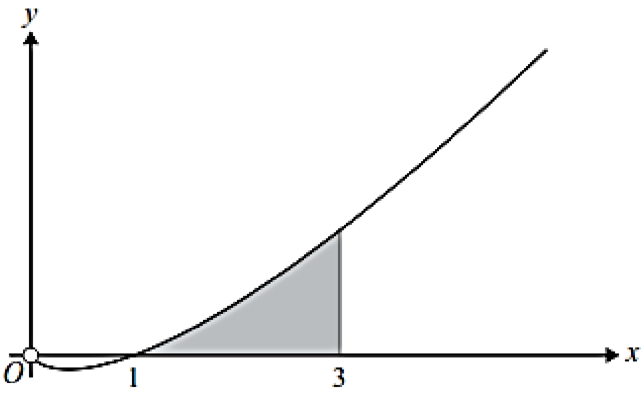
Question 5

The area of the region bounded by the y-axis, the x-axis, the curve $y = e^{2x}$ and the line $x = C$, where C is a positive real constant, is $\frac{5}{2}$. Find C.

3 marks

Question 9

Part of the graph of $f: R^+ \rightarrow R, f(x) = x\log_e(x)$ is shown below.



a. Find the derivative of $x^2 \log_e(x)$.

1 mark

b. Use your answer to **part a.** to find the area of the shaded region in the form $a \log_e(b) + c$ where a, b and c are non-zero real constants.

Question 9

a. Let $f: R \to R$, $f(x) = x\sin(x)$.

Find $f'(x)$.

1 mark

b. Use the result of **part a.** to find the value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\cos(x)\,dx$ in the form $a\pi + b$.

3 marks

MULTIPLE CHOICE QUESTIONS

Question 15

If $\int_0^5 g(x)dx = 20$ and $\int_0^5 (2g(x) + ax)dx = 90$, then the value of a is

A. 0
B. 4
C. 2
D. −3
E. 1

Question 9

Given that $\frac{d(xe^{kx})}{dx} = (kx+1)e^{kx}$, then $\int xe^{kx}dx$ is equal to

A. $\frac{xe^{kx}}{kx+1} + c$

B. $\left(\frac{kx+1}{k}\right)e^{kx} + c$

C. $\frac{1}{k}\int e^{kx}dx$

D. $\frac{1}{k}\left(xe^{kx} - \int e^{kx}dx\right) + c$

E. $\frac{1}{k^2}\left(xe^{kx} - e^{kx}\right) + c$

Question 8

If $\int_1^4 f(x)dx = 6$, then $\int_1^4 (5 - 2f(x))dx$ is equal to

A. 3
B. 4
C. 5
D. 6
E. 16

Question 20

Consider the transformation T, defined as

$$T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

The transformation T maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

If $\int_0^3 f(x)dx = 5$, then $\int_{-3}^0 g(x)dx$ is equal to

A. 0

B. 15

C. 20

D. 25

E. 30

Question 19

If $f(x) = \int_0^x \left(\sqrt{t^2 + 4}\right) dt$, then $f'(-2)$ is equal to

A. $\sqrt{2}$

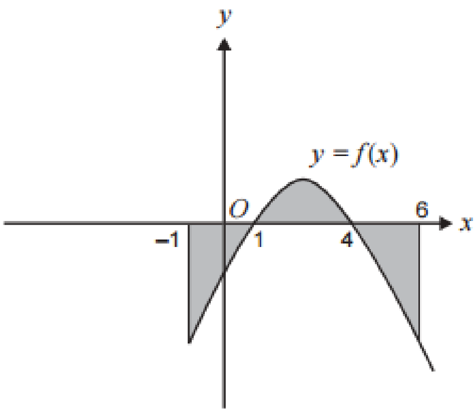
B. $-\sqrt{2}$

C. $2\sqrt{2}$

D. $-2\sqrt{2}$

E. $4\sqrt{2}$

Question 15



The total area of the shaded regions in the diagram is given by

A. $\int_{-1}^{6} f(x)dx$

B. $-\int_{-1}^{0} f(x)dx + \int_{0}^{6} f(x)dx$

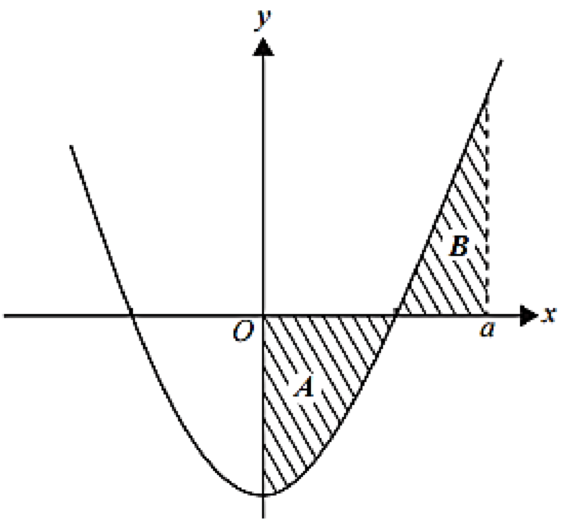
C. $\int_{1}^{4} f(x)dx + 2\int_{4}^{6} f(x)dx$

D. $-\int_{-1}^{1} f(x)dx + \int_{1}^{4} f(x)dx - \int_{4}^{6} f(x)dx$

E. $-\int_{1}^{-1} f(x)dx + \int_{1}^{4} f(x)dx - \int_{6}^{4} f(x)dx$

Question 20

A part of the graph of $g: R \to R, g(x) = x^2 - 4$ is shown below.



The area of the region marked A is the same as the area of the region marked B.

The exact value of a is

A. 0

B. 6

C. $\sqrt{6}$

D. 12

E. $2\sqrt{3}$

Question 20

Let f be a differentiable function defined for all real x, where $f(x) \geq 0$ for all $x \in [0, a]$.

If $\int_0^a f(x)dx = a$, then $2\int_0^{5a} \left(f\left(\frac{x}{5}\right) + 3 \right) dx$ is equal to

A. $2a + 6$

B. $10a + 6$

C. $20a$

D. $40a$

E. $50a$