ANSWERS

A biased die with six faces is rolled. The discrete random variable X represents the score on the uppermost face. The probability distribution of X is shown in the table below.

x	1	2	3	4	5	6
$P(X = x)$	a	a	a	b	b	0.3

(a) Given that $E(X) = 4.2$ find the value of a and the value of b. **(5)**

(b) Show that $E(X^2) = 20.4$ **(1)**

(c) Find: $\Pr(2 < x \leq 5)$

(d) Calculate: $\Pr(2 < x \leq 5 | x > 3)$

(e) Calculate the variance of X.

(f) Calculate the standard deviation of X, correct to two decimal places.

(g) Calculate: $\Pr(\mu - \sigma < x < \mu + \sigma)$

(h) Let $Y = 5 - 2X$. Calculate:

1) $E(Y)$

2) $\mathrm{var}(Y)$

The die above is rolled twice. Let the random variable S be the sum of the two scores that are obtained.

(i) Find the probability distribution of S.

(j) What is the median of S?

(a) $3a + 2b + 0.3 = 1$

$\therefore 3a + 2b = 0.7$ (1) (since all the probabilities must add up to 1)

$1a + 2a + 3a + 4b + 5b + 6 \times 0.3 = 4.2$

$a + 2a + 3a + 4b + 5b + 1.8 = 4.2$ (since we are given E(X))

$6a + 9b = 2.4$

$\therefore 2a + 3b = 0.8$ (2)

$$① \times 2 \quad 6a + 4b = 1.4$$

$$② \times 3 \quad 6a + 9b = 2.4$$

$$\therefore 5b = 1 \qquad \therefore b = 0.2$$

Sub. for b in ①: $3a + 0.4 = 0.7$

$$\therefore a = 0.1$$

$$\therefore b = 0.2, \; a = 0.1$$

(b) $E(X^2) = 1^2 \times 0.1 + 2^2 \times 0.1 + 3^2 \times 0.1 + 4^2 \times 0.2 + 5^2 \times 0.2 + 6^2 \times 0.3$

$$= 0.1 + 0.4 + 0.9 + 3.2 + 5 + 10.8$$

$$= 20.4$$

(c) $Pr(2 < X \leq 5) = Pr(3 \leq X \leq 5)$

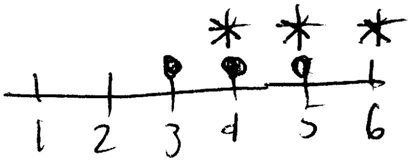
$$= Pr(X=3) + Pr(X=4) + Pr(X=5)$$

$$= 0.1 + 0.2 + 0.2$$

$$= 0.5$$

Conditional probability

(d) $Pr(2 < X \leq 5 \mid X > 3) = \dfrac{Pr(2 < X \leq 5 \cap X > 3)}{Pr(X > 3)}$



$$= \frac{Pr(X=4) + Pr(X=5)}{Pr(X>3)}$$

To find the intersection, it
is useful to draw a diagram.

$$= \frac{0.2 + 0.2}{0.2 + 0.2 + 0.3} = \frac{4}{7}$$

(e) $\boxed{Var(X) = E(X^2) - [E(X)]^2}$ Formula for variance

$$= 20.4 - (4.2)^2$$

$$= 2.76$$

(f) $\sigma = \sqrt{2.76} \simeq 1.66$

standard deviation is the square root of the variance.

(g) $Pr(4.2 - 1.66 < X < 4.2 + 1.66)$

$$= Pr(2.54 < X < 5.86)$$

$$= Pr(3 \leq X \leq 5)$$

$$= 0.5$$

Must find the values of the discrete random variable that lie within these limits.

(h) $E(Y) = 5 - 2E(X)$

$$= 5 - 2 \times 4.2$$

$$= 5 - 8.4$$

$$= -3.4$$

If a and b are constants:

$E(aX+b) = aE(X) + b$

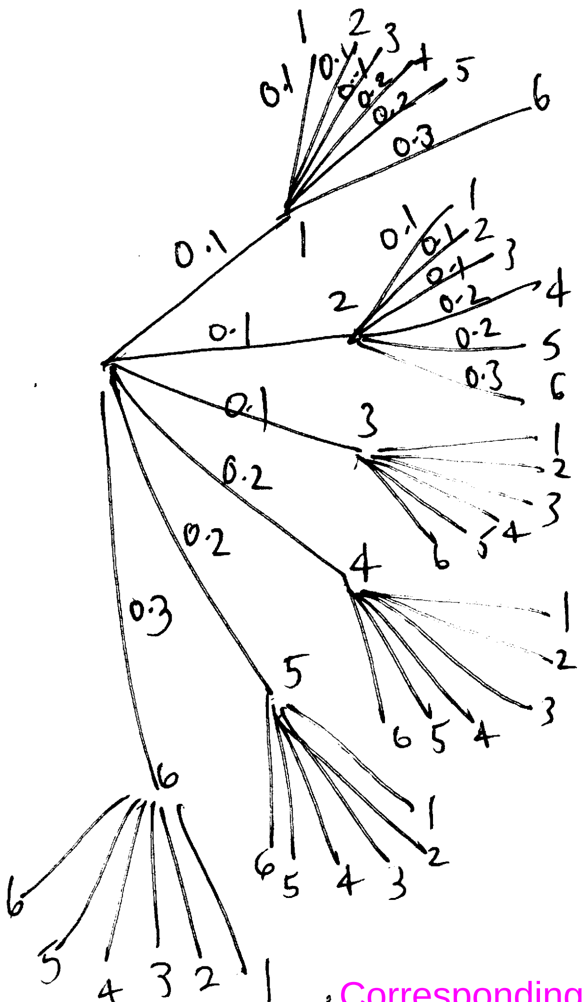
$Var(Y) = Var(5 - 2X)$

$$= (-2)^2 Var(X)$$

$$= 4 \times 2.76$$

$$= 11.04$$

$var(aX + b) = a^2 var(X)$



This question is asking us to construct a table which gives the probability of the random variable S taking its various values.

The only possible values of the random variable S are:

$S = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

Values of S	Corresponding outcomes	Probability that S has this value
$S=2$	$(1,1)$	$(0.1)^2 = \underline{0.01}$
$S=3$	$(2,1), (1,2)$	$(0.1)^2 + (0.1)^2 = \underline{0.02}$
$S=4$	$(1,3), (2,2), (3,1)$	$(0.1)^2 + (0.1)^2 + (0.1)^2 = \underline{0.03}$
$S=5$	$(1,4), (3,2), (2,3), (4,1)$	$0.1 \times 0.2 + (0.1)^2 + (0.1)^2 + 0.2 \times 0.1 = \underline{0.06}$
$S=6$	$(1,5), (2,4), (3,3), (4,2), (5,1)$	$0.1 \times 0.2 + 0.1 \times 0.2 + (0.1)^2 + 0.2 \times 0.1 + 0.2 \times 0.1 = \underline{0.09}$
$S=7$	$(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$	$0.1 \times 0.3 + 0.1 \times 0.2 + 0.1 \times 0.2 + 0.2 \times 0.1 + 0.2 \times 0.1 + 0.3 \times 0.1 = \underline{0.14}$

	Outcomes	
$S=8$	$(2,6),(3,5),(4,4),(5,3),(6,2)$	$0.1\times0.3+0.1\times0.2+(0.2)^2+0.2\times0.1+0.3\times0.1 = \underline{0.14}$
$S=9$	$(3,6),(4,5),(5,4),(6,3)$	$0.1\times0.3+0.2\times0.2+0.2\times0.2+0.3\times0.1 = \underline{0.14}$
$S=10$	$(4,6),(5,5),(6,4)$	$0.2\times0.3+0.2\times0.2+0.3\times0.2 = \underline{0.16}$
$S=11$	$(5,6),(6,5)$	$0.2\times0.3+0.3\times0.2 = \underline{0.12}$
$S=12$	$(6,6)$	$(0.3)^2 = \underline{0.09}$

Probability distribution of S

S	2	3	4	5	6	7	8	9	10	11	12
$Pr(S=s)$	0.01	0.02	0.03	0.06	0.09	0.14	0.14	0.14	0.16	0.12	0.09

(ii) $Pr(S \le 8) = 0.49$

$Pr(S \le 9) = 0.63$

$\therefore$ $S = 9$ is the median.

(The median is that value of the discrete random variable for which there is a probability of 0.5 that the random variable is less than or equal to it. However, if we can't get to 0.5 exactly, we must go to the next value for which there is at least 50% probability that the variable is less than or equal to it.

CLASS QUESTIONS ANSWERS

Question 1

When a student is completing a task set by their teacher on Spacemaths, the number of hints used is monitored by the system.

The probability of using at least 1 hint is 0.8. $\rightarrow Pr(X \geq 1) = 0.8 \quad \therefore Pr(X=0) = 0.2$

The probability of using 2 hints is the same as using 3 hints.

The probability of using 1 hint is the same as using 4 hints. $\rightarrow Pr(X=4) = 0.1$

At most students can use 4 hints.

The probability they use 2 hints is half of the probability that they use 0 hints. $\rightarrow Pr(X=2) = 0.1$

Let X represent the number of hints they used.

Let $Pr(X=2) = Pr(X=3) = p$

Construct the probability distribution. Write the possible values of X in ascending order from left to right.

x	0	1	2	3	4
$P(X=x)$	0.2	0.1	0.3	0.3	0.1

$0.2 + 0.1 + 0.1 + 2p = 1$

$\therefore 2p = 0.6$

$p = 0.3$

(a) What is the expected number of hints a student will use?

(b) Given that a student used at least 2 hints, what is the probability they used 4 hints?

(a) $E(X) = 0 \times 0.2 + 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.3 + 4 \times 0.1$

$= 0.1 + 0.6 + 0.9 + 0.4$

$= 2.0$

(b) $Pr(X=4 \mid X \geq 2) = \dfrac{Pr(X=4 \cap X \geq 2)}{Pr(X \geq 2)} = \dfrac{Pr(X=4)}{Pr(X \geq 2)}$

$= \dfrac{0.1}{0.7} = \dfrac{1}{7}$

Question 2

At a car park in the city, all day parking is charged on the following basis:

- Cars with just a driver pay \$20
- Cars with a driver and one passenger pay \$18
- Cars with a driver and at least two passengers pay \$15

The number of people in one of these cars on a given day is summarised in the table.

Number of people	1	2	3	4	5
Number of cars	4 600	3 400	1 100	700	200

P = no. of people in car

Total no. of Cars $= 4600 + 3400 + 1100 + 700 + 200 = 10{,}000$

Calculate the probability a randomly selected car is carrying 3 people.

$$\Pr(P=3) = \frac{1100}{10000} = 0.11$$

Given that a car was carrying at least 2 people, what is the probability it was carrying 4?

$$\Pr(P=4 \mid P \geq 2) = \frac{\Pr(P=4)}{\Pr(P\geq 2)}$$

$$= \frac{\frac{700}{10000}}{\frac{5400}{10000}} = \frac{700}{5400} = \frac{7}{54}$$

Let X represent the parking fee paid by a randomly selected car. Construct the probability distribution for X below.

Write the possible values of X in descending order from left to right.

x	20	18	15
$P(X=x)$	0.46	0.34	0.2

Calculate the expected revenue per car in this car park.

$$E(X) = 0.46\times 20 + 0.34\times 18 + 15\times 0.2$$

$$= \$18.32$$

Question 3

The table below represents a discrete probability distribution.

x	1	2	3	4	5
$P(X=x)$	0.05	m	0.25	n	0.1

(a) Use a property of probability distributions to express m in terms of n.

$0.05+m+0.25+n+0.1=1$

$\therefore \quad 0.4+m+n=1$

$\therefore m=0.6-n$

(b) Use the fact that $E(X)=3.3$ to express m in terms of n.

(c) Hence solve for n.

(d) Hence solve for m.

Calculate the standard deviation of the distribution.

Give your answer to one decimal place.

(b) $E(X)=3.3 \qquad \therefore 3.3=1\times0.05+2m+3\times0.25+4n+5\times0.1$

$3.3 = 0.05+2m+0.75+4n+0.5$

$3.3 = 1.3+2m+4n$

$2 = 2m+4n$

$\therefore 1 = m+2n \qquad \therefore m=1-2n$

(c) $1-2n=0.6-n$

$\therefore 0.4=n$

(d) $m = 1-2\times0.4 = 0.2$

(e) $E(X^2)=1^2\times0.05+2^2\times0.2+3^2\times0.25+4^2\times0.4+5^2\times0.1$

$=12$

$\therefore var(X)=E(X^2)-[E(X)]^2 = 12-3.3^2 = 1.11$

$\therefore \sigma = \sqrt{1.11} \approx 1.1$