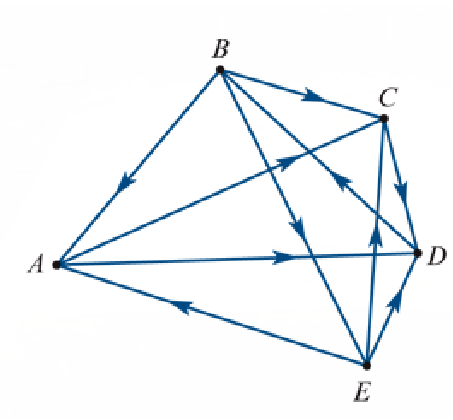
Dominance Matrices

In many groups of individuals, animals, ecosystems, there is a definite “pecking order” or hierarchy where one member of the group dominates another. An example from sport is a round robin tournament such as tennis where no ties are permitted and each player plays every other player once. Or a soccer tournament where the same conditions apply.

So for example, suppose there is a tennis tournament wit 5 players: A, B, C, D, E and the following results occur:

A defeats C and D; B defeats A, C and E; C defeats D ; D defeats B ; E defeats A, C and D.

We can represent these tournament results as a network diagram graphically. An arrow from vertex B to vertex A means that B defeated A:



NOTE: Arrows can only go one way because players only play each other once.

We can represent this network diagram as a one-step Domination matrix ***D***:

to

$$D = \text{from} \quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$

This is a **one step Domination** matrix because it tells us who defeated who, or who dominated over who, in their individual game. If we add the elements in each row, we get the total number of wins (one-step dominances) of each player.

(The total of each column is the total number of losses for each player).

Now we can also take into account two step dominances in order to rank the players. A two step dominance occurs when for example A defeats C who then defeats D. A is said to have a **two step dominance** over D.

The two step dominance matrix is found by evaluating D^2 :

$$\text{From}\;\begin{matrix} & \overset{\text{to}}{\begin{matrix} A & B & C & D & E \end{matrix}} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix} \times \begin{matrix} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

B def A

B def C

B def E

A def D

C def D

E def D

$$D^2 = \underset{\text{from}}{\begin{matrix} & \overset{\text{to}}{\begin{matrix} A & B & C & D & E \end{matrix}} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} & \begin{bmatrix} * & * & * & * & * \\ * & * & * & \boldsymbol{p} & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix} \end{matrix}}$$

The value of *p* is found by multiplying row B by column D:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 1 \times 1 + 0 \times 0 + 1 \times 1 + 0 \times 0 + 1 \times 1 = 3$$

because: B def A and A def D

B def C and C def D

B def E and E def D

B has a 3 two step dominances over D

When calculated we get:

to

from

$$D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Two step dominance scores are found from

adding the elements in each row:

$$D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} Dominance \\ 2 \\ 6 \\ 1 \\ 3 \\ 4 \end{array}$$

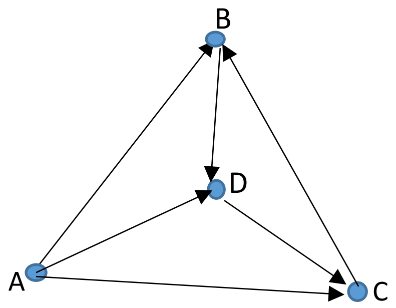
Now we can rank the players by taking into account one-step and two-step dominances by calculating a new matrix: $T = D + D^2$

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} + \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array} =$$

Based on $\boldsymbol{T} = \boldsymbol{D} + \boldsymbol{D}^{2}$ where *T* is the total matrix of one and two step dominances, we get the final rankings in the tournmament:

Question 1

For the network diagram shown below, construct *D* and calculate D^2. Calculate *T* and hence rank the dominance of each vertex.



Question 2

Five soccer teams, A, B, C , D and E play a tournament where no draws are allowed (if necessary games are decided by penalty shoot outs). The results were:

A defeats B, C and D

B defeats C and E

C defeats D and E

D defeats B

E defeats A and D

a. Draw a diagram to reflect these results and construct the dominance matrix D.
b. Calculate $T = D^2 + D$
c. Rank the teams according to the matrix T.