2013 Mathematical Methods (CAS) GA 2: Examination 1

GENERAL COMMENTS

The 2013 Mathematical Methods (CAS) examination 1 focused on the topics of calculus (differentiation, anti-differentiation, definite integrals, area between curves, rule of function from derivative, average value of a function, mean of a continuous probability density function), functions (quadratic, absolute value, circular, exponential, sketch graphs, transformations, domain, points of intersection), probability (total probability, determination of E(X) for discrete and continuous distributions), index and logarithm laws.

Within these topics it was important that students were adept at algebraic manipulation and arithmetic computation (operations with integers, decimals and fractions). Algebraic manipulation was prominent in Questions 3, 5a., 5b., 6, 7a., 8, 9a., 9ci., 10a. and 10b. Arithmetic was a problem for some students in Questions 1b., 3, 5b., 7a., 7bi. and 7bii. Final responses to questions should be given in a simplified form where there are simple common factors.

In the 2012 exam report, it was stated that 'when a function has been changed through transformations, then it is no longer the same function'. However, many students did not heed this advice in Question 9ci. and called their image function $g(x)$.

Solving quadratic equations was necessary for Questions 6, 7a. and 9a. In each case, once the quadratic was established, the majority of students completed the question successfully. Students needed to explicitly state the relevant equation and apply the null factor law if appropriate (the null factor law was also a feature of Question 10b.). It was disappointing to see the number of students who attempted to solve a quadratic equation in the form $ax^2 + bx = c$.

READ THE PARAGRAPH IN THE BOX BELOW!!

To be successful in this examination, students needed to know and be able to use exact values for sine and cosine functions, logarithm and index laws, the average value of a function, the quadratic formula or how to complete the square, definite integrals, area between curves, and know that e^{kx} is a positive for all real values of x.

The majority of students were able to complete the paper within the allocated time. Students should make good use of the 15 minutes of reading time and ensure they understand the questions before they begin writing. Students could try to identify questions that have familiar concepts and routines, and start with those when writing time begins.

Students should also detach the sheet of miscellaneous formulas during reading time.

SPECIFIC INFORMATION

This report provides sample answers or an indication of what answers may have included. Unless otherwise stated, these are not intended to be exemplary or complete responses.

The statistics in this report may be subject to rounding errors resulting in a total less than 100 per cent.

Question 1a.

Marks	0	1	2	Average
%	10	12	78	**1.7**

DON' T READ ANY FURTHER UNTIL YOU HAVE C

$$\frac{dy}{dx} = x^2 \times \frac{1}{x} + 2x\log_e(x) = x + 2x\log_e(x)$$

Some students did not simplify the expression or incorrectly combined the terms to obtain $3x\log_e(x)$.

Question 1b.

Marks	0	1	2	3	Average
%	5	23	13	58	**2.3**

$$f'(x) = e^{x^2} \times 2x$$

$$f'(3) = e^9 \times 6 = 6e^9$$

This question was generally well done. Some students neglected a substitution, whereas some substituted incorrectly. A common error was to interpret e^{x^2} as $(e^x)^2$.

Question 2

Marks	0	1	2	Average
%	24	26	50	1.3

An antiderivative is $\frac{(4-2x)^{-4}}{-4\times -2}=\frac{(4-2x)^{-4}}{8}$ or $\frac{1}{8(4-2x)^4}$.

The most common error with this question was neglecting to divide by the coefficient of *x*.

Question 3

Marks	0	1	2	Average
%	26	32	42	1.2

$g(x)=\frac{-\cos(2\pi x)}{2\pi}+c$

$g(1)=\frac{-\cos(2\pi)}{2\pi}+c=\frac{1}{\pi}$, as $\cos(2\pi)=1$

$c=\frac{1}{\pi}+\frac{1}{2\pi}=\frac{3}{2\pi}$

Answer: $\text{g}(x)=\frac{-\cos(2\pi x)+3}{2\pi}$

Most students could anti-differentiate $\sin(2\pi x)$ but many neglected '+ *c*', which was essential in order to move to the next step. Mistakes in the attempt to combine the two fractions for *c* were common.

Question 4

Marks	0	1	2	Average
%	23	31	47	1.3

$\frac{x}{2}=\frac{7\pi}{6},\frac{11\pi}{6}$

$\Rightarrow x=\frac{7\pi}{3},\frac{11\pi}{3}$

This question was generally well done. Many students identified a base angle of $\frac{\pi}{6}$ but many could not identify the correct quadrants and domain restriction.

Question 5a.

Marks	0	1	2	Average
%	13	32	54	1.4

$\log_3\left(\frac{25x}{2}\right)=2$

$\frac{25x}{2}=9 \quad \Rightarrow x=\frac{18}{25}$

Many students solved this equation correctly. A disappointing number of students could not combine all parts of the logarithms into a single expression.

Question 5b.

Marks	0	1	2	Average
%	14	23	64	1.5

$9^{6-x}=3^{2(6-x)}\Rightarrow 3^{-4x}=3^{2(6-x)}$

$\Rightarrow -4x=12-2x \quad \Rightarrow x=-6$

The majority of incorrect responses involved $9 = 3^3$.

2013 Examination Report

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Question 6

Marks	0	1	2	3	Average
%	40	19	25	16	1.2

$$\frac{1}{2}\int_{-1}^{1} g(x)dx = \frac{31}{12}$$

$$\Rightarrow \left[\frac{(a-x)^3}{-3}\right]_{-1}^{1} = \frac{31}{6} \qquad \text{or alternatively } \left[a^2x - ax^2 + \frac{x^3}{3}\right]_{-1}^{1} = \frac{31}{6}$$

$$\Rightarrow \left[(a-1)^3 - (a+1)^3\right] = \frac{-31}{2} \qquad \Leftrightarrow \left[(a^2 - a + \frac{1}{3}) - (-a^2 - a - \frac{1}{3})\right] = \frac{31}{6}$$

$$\left[a^3 - 3a^2 + 3a - 1 - (a^3 + 3a^2 + 3a + 1)\right] \qquad \Leftrightarrow [2a^2 + \frac{2}{3}] = \frac{31}{6}$$

$$\Rightarrow [-6a^2 - 2] = \frac{-31}{2}$$

$$\Rightarrow a^2 = \frac{27}{12} = \frac{9}{4}$$

$$\therefore a = \pm\frac{3}{2}$$

Make sure that you do NOT confuse the average value of a function with "average rate of change", or "expectation" or "average of". The average value of a function has a specific meaning

Most students attempted this question but far too many misunderstood 'average value' to be either 'average rate' or 'average of'. The correct anti-derivative of the bracketed form of $g(x)$ was less evident than for the expanded form, due to the necessary division by the coefficient of x.

The formula for the 'average value' was not on the formula sheet. Students should ensure that they have a good understanding of the concept that the average value of a function f over the interval $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

Question 7a.

Marks	0	1	2	3	Average
%	7	19	28	46	2.2

$$\sum \Pr(X = x) = 0.2 + 0.6p^2 + 0.1 + 1 - p + 0.1 = 1$$

$$0.6p^2 - p + 0.4 = 0 \quad \Rightarrow \quad p = \frac{1 \pm \sqrt{1 - 0.96}}{1.2} = \frac{1 \pm 0.2}{1.2} = 1 \text{ or } \frac{0.8}{1.2} = \frac{2}{3}$$

or $6p^2 - 10p + 4 = 0 \Rightarrow 3p^2 - 5p + 2 = 0$

$$\Rightarrow (3p-2)(p-1) = 0 \quad \therefore p = \frac{2}{3} \text{ or } p = 1$$

Note the strategy here of multiplying through by a common factor (10 in this case) to make the coefficients whole numbers.

This question presented a range of fundamental problems for students: mixed operations with fractions and decimals, factorising and solving quadratics, using substitution to 'show' and not eliminate or determine other possible solutions, and poor notation.

Question 7bi.

Marks	0	1	2	Average
%	9	62	29	1.2

$$E(X) = \sum x\Pr(X = x) = 1 \times 0.6 \times \frac{4}{9} + 0.2 + 3(1 - \frac{2}{3}) + 0.4 = \frac{3^1}{5} \times \frac{4}{9_3} + \frac{1}{5} + 1 + \frac{2}{5} = 1 + \frac{13}{15}$$

$$E(X) = 1\frac{13}{15} \text{ or } \frac{28}{15} \text{ or } 1.8\dot{6}$$

Many students had difficulty adding decimals and fractions to give an answer.

Question 7bii.

Marks	0	1	Average
%	68	32	0.3

$$\Pr(X \ge E(X)) = \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4) = \frac{1}{10} + \frac{1}{3} + \frac{1}{10} = \frac{3+10+3}{30} = \frac{16}{30} = \frac{8}{15}$$

Many students gave 0.5 as the answer.

Question 8

Marks	0	1	2	3	Average
%	47	20	12	21	1.1

$$E(X) = \int_0^2 x \times \frac{\pi}{4}\cos\left(\frac{\pi x}{4}\right)dx$$

$$\Rightarrow E(X) = \int_0^2 \left(\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right) - \sin\left(\frac{\pi x}{4}\right)dx\right.$$

$$= \left[x\sin\left(\frac{\pi x}{4}\right)\right]_0^2 - \left[\frac{-4}{\pi}\cos\left(\frac{\pi x}{4}\right)\right]_0^2 = [2\sin\left(\frac{\pi}{2}\right) - 0] + \left[\frac{4}{\pi}\cos\left(\frac{\pi}{2}\right) - \frac{4}{\pi}\cos(0)\right]$$

$$= 2 + \left[-\frac{4}{\pi}\right]$$

$$\therefore E(X) = 2 - \frac{4}{\pi}$$

This question challenged students. As $E(X) = \int_0^2 xf(x)dx$, the key was to recognise that the given relation for $\frac{d}{dx}\left(x\sin\left(\frac{\pi x}{4}\right)\right)$ provided the required anti-derivative. Care then needed to be taken with substitution and evaluation.

Question 9a.

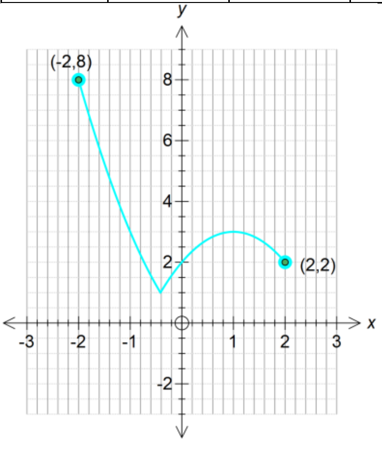
Marks	0	1	Average
%	50	50	0.5

$a = 1 - \sqrt{2}$

The most common errors included not giving the answer in the form required, choosing the alternative *x*-value and not setting up the correct quadratic.

Question 9b.

Marks	0	1	2	Average
%	39	15	46	1.1



Some students drew the correct graph but did not have correct endpoints; some had correct endpoints but were not careful enough with the placement of the other key features. Most students knew what the shape would be but had the graph in the incorrect location.

Question 9ci.

Marks	0	1	2	Average
%	20	64	16	**1**

Rule for *image* of g, $g_1(x)=\frac{1}{3}(g(x+1)-1)=\frac{1}{3}\left|x^2-2\right|$. Note the use of a different function name for the image.

The order of transformations may have been different from what many students had experienced and proved to be the stumbling block for most.

Students who used matrices to establish their equation rarely completed the translation first. The required matrix equation was $\begin{bmatrix} x' \\ y' \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right)$

It is important that the original function and image functions have distinct names.

Question 9cii.

Marks	0	1	Average
%	61	39	**0.4**

$x \in [-3,1]$

The frequency of [–2,2] as a preferred solution raises the concern that students believe that the domain does not change under transformations. They may have confused this with rules applied to either composite functions or the addition, subtraction and multiplication of functions.

Question 10a.

Marks	0	1	Average
%	45	55	**0.6**

$A=\frac{1}{2} base \times height=\frac{1}{2}x\times 2e^{\frac{-x}{5}}=xe^{\frac{-x}{5}}$

Most students knew how to answer this question. Many students left the equation unsimplified.

Question 10b.

Marks	0	1	2	3	Average
%	46	11	17	26	**1.2**

$\frac{dA}{dx}=e^{\frac{-x}{5}}+x\times e^{\frac{-x}{5}}\times\frac{-1}{5}=e^{\frac{-x}{5}}-\frac{1}{5}xe^{\frac{-x}{5}}=e^{\frac{-x}{5}}\left(1-\frac{x}{5}\right)$

$\frac{dA}{dx}=0$ for max/min, $x=5$ (as $e^{\frac{-x}{5}}>0$)

$\therefore$ max Area $=\frac{5}{e}$

It was necessary to have a result in part a. in order to make any progress in this question. The majority of students who obtained two marks produced additional incorrect solutions when equating the derivative to zero.

Question 10c.

Marks	0	1	2	3	Average
%	46	20	27	7	1

Location of x value at T: $f(x) = 2e^{\frac{-x}{5}} = \frac{1}{2} \Rightarrow e^{\frac{-x}{5}} = \frac{1}{4} \Rightarrow \frac{-x}{5} = \log_e(\frac{1}{4})$

$\therefore x = -5\log_e(\frac{1}{4})$ or $5\log_e(4)$ or $10\log_e(2)$

Area of sliver = Area of trapezium under ST $- \int_0^{5\log_e(4)} 2e^{\frac{-x}{5}}\,dx$ or $\int_0^{5\log_e(4)} \frac{3x}{10\log_e(4)} + 2 - 2e^{\frac{-x}{5}}\,dx$

$= \frac{\frac{1}{2}+2}{2} \times 5\log_e(4) - \left[-10e^{\frac{-x}{5}}\right]_0^{5\log_e(4)} = \frac{25}{4}\log_e(4) + \left[(10e^{\frac{-5\log_e(4)}{5}}) - (10e^0)\right] = \frac{25}{4}\log_e(4) + \frac{10}{4} - 10$

$Answer = \frac{25}{4}\log_e(4) - \frac{15}{2}$

Many students made a start with this question, yet most had difficulty with the substitutions required to continue. Students who obtained two marks neglected to take into account the rectangle below the $y = 0.5$ line.