Q10.

(a) $f(x) = \dfrac{ax+b}{cx+d}$

$cx+d \neq 0$

$x \neq -\dfrac{d}{c}$

$$\require{enclose}\begin{array}{r} \frac{a}{c} \\ cx+d \enclose{longdiv}{ax+b} \\ -\left(ax+\frac{da}{c}\right) \\ \hline b-\frac{da}{c} \end{array}$$

$\therefore f(x) = \dfrac{a}{c} + \dfrac{b-\frac{da}{c}}{cx+d}$

$\therefore \operatorname{ran}(f) = \mathbb{R}\setminus\left\{\frac{a}{c}\right\}$

dom(f)	ran(f)
$\mathbb{R}\setminus\left\{-\frac{d}{c}\right\}$	$\mathbb{R}\setminus\left\{\frac{a}{c}\right\}$
dom(f^{-1})	ran(f^{-1})
$\mathbb{R}\setminus\left\{\frac{a}{c}\right\}$	$\mathbb{R}\setminus\left\{-\frac{d}{c}\right\}$

Now interchange x and y to find the rule:

$y = \dfrac{ax+b}{cx+d}$

$\therefore x = \dfrac{ay+b}{cy+d}$

$cyx + dx = ay + b$

$\therefore cyx - ay = b - dx$

$\therefore y(cx-a) = b - dx$

$y = \dfrac{b-dx}{cx-a}$

$\therefore f^{-1}(x) = \dfrac{b-dx}{cx-a}, \quad x \in \mathbb{R}\setminus\left\{\frac{a}{c}\right\}$

Q10.(b) Just substitute for a, b, c, d into rule for $f^{-1}(x)$ found in Q10(a):

(i) If $a = 3, b = 2, c = 3, d = 1$

$$f^{-1}(x) = \frac{2 - x}{3x - 3}, \quad x \in R \setminus \left\{\frac{3}{3}\right\}$$

$$\therefore f^{-1}(x) = \frac{2 - x}{3(x - 1)}, \quad x \in R \setminus \{1\}$$

(ii) $a = 3, b = 2, c = 2, d = -3$

$$f^{-1}(x) = \frac{2 - -3x}{2x - 3}, \quad x \in R \setminus \left\{\frac{3}{2}\right\}$$

$$\therefore f^{-1}(x) = \frac{2 + 3x}{2x - 3}, \quad x \in R \setminus \left\{\frac{3}{2}\right\}$$

(iii) $a = 1, b = -1, c = -1, d = -1$

$$\therefore f^{-1}(x) = \frac{-1 - -x}{-x - 1}, \quad x \in R \setminus \left\{\frac{1}{-1}\right\}$$

$$\therefore f^{-1}(x) = \frac{x - 1}{-x - 1}, \quad x \in R \setminus \{-1\}$$

(iv) $a = -1, b = 1, c = 1, d = 1$

$$\therefore f^{-1}(x) = \frac{1 - x}{x - -1}, \quad x \in R \setminus \left\{\frac{-1}{1}\right\}$$

$$\therefore f^{-1}(x) = \frac{1 - x}{x + 1}, \quad x \in R \setminus \{-1\}$$

Q 10 (c)

If $f(x) = f^{-1}(x)$ for all values of x then:

$$\frac{ax+b}{cx+d} = \frac{b-dx}{cx-a}$$

$\therefore (ax+b)(cx-a) = (cx+d)(b-dx)$

$\therefore acx^2 - a^2x + bcx - ba = bcx - cdx^2 + db - d^2x$

$\therefore x^2(ac+cd) + x(d^2-a^2) - ba - db = 0.$

Now this equation must always be true, whatever the value of x.

Therefore:

$ac + cd = 0 \quad \therefore a = -d \quad (1)$

$d^2 - a^2 = 0 \quad \therefore a = \pm d \quad (2)$

$-(ba+db) = 0 \quad \therefore ba + db = 0 \quad (3)$

For (1) and (2) both to be true, we require: $a = -d$.

If $a = -d$ then (3) will be true, since if we substitute: $a = -d$ in:

$ba + db$

$= -bd + db = 0.$

Therefore, $f = f^{-1}$ if $a = -d$.