Section C (continued)

For Marker Use Only

Question 31

(4 marks)

(a) The function $h(x) = px^3 - qx^2 + x$ has a stationary point at $(-2, 0)$. Determine the values of the constants p and q.

$h(-2) = 0$

$h'(-2) = 0$

Solving: $p = \frac{1}{4}$, $q = -1$

(b) Hence, solve $h'(x) = 0$ to determine the **nature** and **exact location** for the other stationary point.

$h(x) = \frac{1}{4}x^3 + x^2 + x$

$h'(x) = \frac{3}{4}x^2 + 2x + 1$

$h'(x) = 0$ if $x = -2, -\frac{2}{3}$

$h(-2) = 0$, $h\left(\frac{-2}{3}\right) = \frac{-8}{27}$

x	-3	-2	-1	$\frac{-2}{3}$	0
$h'(x)$	$\frac{7}{4}$	0	$-\frac{1}{4}$	0	1
	/	—	\	—	/

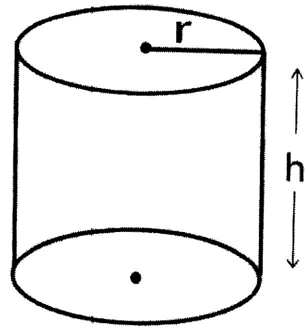
∴ Max t/p at: $(-2, 0)$, Min t/p at $\left(\frac{-2}{3}, \frac{-8}{27}\right)$

Section C continues over the page.

Section C (continued)

For Marker Use Only

Question 32



A soft drink company is reviewing the dimensions of their 325 mL aluminium cans. The price of aluminium is predicted to rise and the company directors are keen to reduce costs by minimising the surface area.

Given that $V = \pi r^2 h$ and $A = 2\pi r^2 + 2\pi r h$ and $1 \text{ mL} = 1 \text{ cm}^3$, where V=volume (325 cm^3), r = radius (cm), h = height (cm) and A = surface area (cm^2),

(6 marks)

(a) Show that $A = 2\pi r^2 + 650r^{-1}$.

$325 = \pi r^2 h \quad \therefore h = \dfrac{325}{\pi r^2}$

$A = 2\pi r^2 + 2\pi r h$

$\therefore A = 2\pi r^2 + 2\pi r\left(\dfrac{325}{\pi r^2}\right)$

$\therefore A = 2\pi r^2 + \dfrac{650}{r}$

(b) Hence determine an expression for $\dfrac{dA}{dr}$.

$A = 2\pi r^2 + 650r^{-1}$

$\therefore \dfrac{dA}{dr} = 4\pi r - 650r^{-2}$

$\therefore \dfrac{dA}{dr} = 4\pi r - \dfrac{650}{r^2}$

(c) Find the **dimensions** of the can and **minimum surface area** possible for 325 mL cans. Express answers accurate to 2 decimal places and provide reasoning as to why these dimensions lead to a minimum surface area.

This means that we need to prove it is a minimum using the Gradient chart

For a minimum, $\dfrac{dA}{dr} = 0$

$\therefore 4\pi r - \dfrac{650}{r^2} = 0$

$\therefore 4\pi r^3 = 650$

$r^3 = \dfrac{650}{4\pi} \quad \therefore r = \sqrt[3]{\dfrac{650}{4\pi}} \approx 3.73 \text{ cm}$

$h = \dfrac{325}{\pi r^2}$ When $r = \sqrt[3]{\dfrac{650}{4\pi}}$, $h \approx 7.45 \text{ cm}$

r	2	3.73	4
$A'(r)$	-137	0	9.64
	\	—	/

At $r = 3.73$, there is a minimum t/p.

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

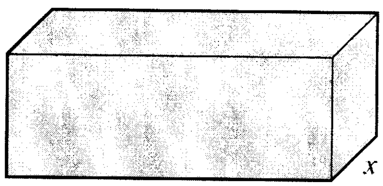
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

A solid block in the shape of a rectangular prism has a base of width x cm. The length of the base is two-and-a-half times the width of the base.



The block has a total surface area of 6480 sq cm.

a. Show that if the height of the block is h cm, $h = \dfrac{6480 - 5x^2}{7x}$.

$$6480 = 2 \times \left(\frac{5x}{2} \times h\right) + 2 \times (hx) + 2 \times \left(\frac{5x}{2}\right)(x)$$

$$6480 = 2 \times \left(\frac{5xh}{2}\right) + 2hx + 2 \times \frac{5x^2}{2}$$

$$6480 = 5xh + 2hx + 5x^2$$

$$\therefore 6480 = 7xh + 5x^2$$

$$\therefore \frac{6480 - 5x^2}{7x} = h$$

2 marks

b. The volume, V cm^3, of the block is given by $V(x) = \frac{5x(6480 - 5x^2)}{14}$.
Given that $V(x) > 0$ and $x > 0$, find the possible values of x. In other words, they want the domain.

$$6480 - 5x^2 > 0$$
$$\therefore 6480 > 5x^2$$
$$\therefore x < \sqrt{\frac{6480}{5}}$$
$$\therefore x < 36$$
$$\therefore x \in (0, 36)$$

2 marks

c. Find $\frac{dV}{dx}$, expressing your answer in the form $\frac{dV}{dx} = ax^2 + b$, where a and b are real numbers.

$$V(x) = \frac{16200x}{7} - \frac{25x^3}{14}$$
$$\therefore V'(x) = \frac{16200}{7} - \frac{75x^2}{14}$$

3 marks

d. Find the exact values of x and h if the block is to have maximum volume.

For a maximum, $V'(x) = 0$

$$\therefore \frac{16200}{7} - \frac{75x^2}{14} = 0$$
$$\therefore x = 12\sqrt{3} \quad (\text{Since } 0 < x < 36)$$

and $h = \frac{6480 - (12\sqrt{3})^2 \times 5}{7 \times 12\sqrt{3}} = \frac{120\sqrt{3}}{7}$

2 marks

For Marker Use Only

Question 36

The Australian native ground cover plant, Hibbertia Procumbens, can cover the ground at a rate modelled by $\frac{dA}{dt} = 3t + 6t^2 - \frac{1}{5}t^3$ where A is the area in square centimetres and t is time in months after the initial planting.

The model is considered effective over the first 30 months when transplanted from a pot into the ground.

(6 marks)

(a) If the plant initially covers 300 cm^2, find an expression for the area, A, covered by the plant.

$\frac{dA}{dt} = 3t + 6t^2 - \frac{t^3}{5}$

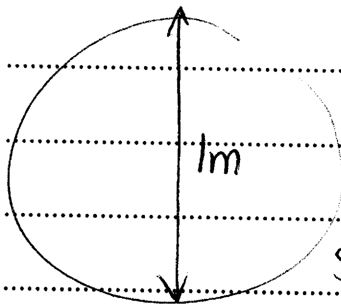
$\therefore A = \int 3t + 6t^2 - \frac{t^3}{5}\, dt$

$= \frac{3t^2}{2} + 2t^3 - \frac{t^4}{20} + c$

$A(0) = 300 \quad \therefore c = 300$

$\therefore A(t) = \frac{3t^2}{2} + 2t^3 - \frac{t^4}{20} + 300$

(b) When the plant reaches maturity, it has a ground covering diameter of 1 metre. Calculate the time, correct to the nearest month, before the Hibbertia is expected to reach full size.



$A = \pi r^2$ where $r = 0.5\text{ m} = 50\text{ cm}$

$A = \pi \times (0.5)^2\text{ m}^2 = (\pi) \times (50)^2\text{ cm}^2$

Solving

$\frac{3t^2}{2} + 2t^3 - \frac{t^4}{20} + 300 = \pi \times (50)^2$

for t: $t = 18.777$ or $t = 38.045$

Since $0 < t < 30$, $t = 18.777$ months

$\simeq 19$ months

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

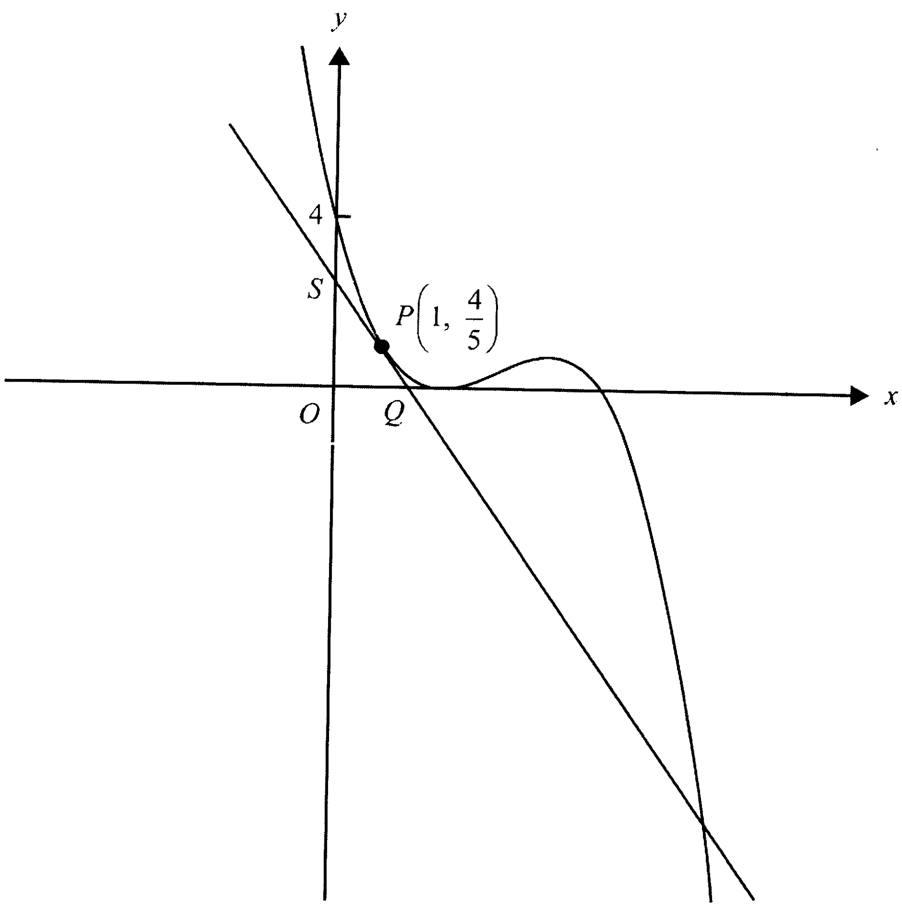
In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (9 marks)

Let $f: R \to R$, $f(x) = \frac{1}{5}(x-2)^2(5-x)$. The point $P\left(1, \frac{4}{5}\right)$ is on the graph of f, as shown below.

The tangent at P cuts the y-axis at S and the x-axis at Q.



a. Write down the derivative $f'(x)$ of $f(x)$. 1 mark

$$f'(x) = \frac{-3(x-4)(x-2)}{5}$$

Use CAS for this. It is only worth 1 mark so no working is expected.

b. **i.** Find the equation of the tangent to the graph of f at the point $P\left(1, \frac{4}{5}\right)$. 1 mark

$$y = \frac{13}{5} - \frac{9x}{5}$$

Use the Tangent line facility on the Calculus menu to get the equation directly from CAS as the question is only worth 1 mark.

ii. Find the coordinates of points Q and S. 2 marks

$$S = \left(0, \frac{13}{5}\right)$$

$$Q = \left(\frac{13}{9}, 0\right)$$

c. Find the distance PS and express it in the form $\frac{\sqrt{b}}{c}$, where b and c are positive integers. 2 marks

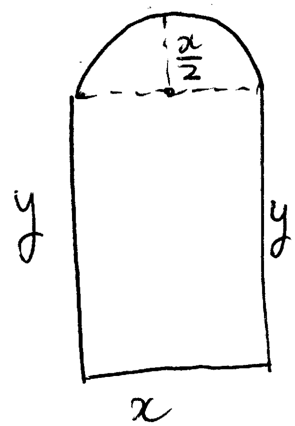
$$P = \left(1, \frac{4}{5}\right)$$

$$S = \left(0, \frac{13}{5}\right)$$

$$d = \sqrt{\left(\frac{13}{5} - \frac{4}{5}\right)^2 + (0-1)^2}$$

$$= \sqrt{\left(\frac{9}{5}\right)^2 + 1}$$

$$= \frac{\sqrt{106}}{5}$$



(i) $$10 = 2x + 2y + \frac{2\pi\left(\frac{x}{2}\right)}{2}$$

$$10 = 2x + 2y + \frac{\pi x}{2}$$

$$\therefore 2y = 10 - 2x - \frac{\pi x}{2}$$

$$y = 5 - x - \frac{\pi x}{4}$$

(ii) L = 1 × Area of semi-circle + 3 × Area of rectangle

$$= \frac{\pi\left(\frac{x}{2}\right)^2}{2} + 3xy$$

$$= \frac{\pi x^2}{8} + 3x\left(5 - x - \frac{\pi x}{4}\right)$$

$$= \frac{\pi x^2}{8} + 15x - 3x^2 - \frac{3\pi x^2}{4}$$

$$L = 15x - 3x^2 - \frac{5\pi x^2}{8}$$

(iii) $L(x) = 15x - x^2\left(3 + \frac{5\pi}{8}\right)$

For a maximum, $L'(x) = 0$

$$\therefore \quad 15 - 2x\left(3 + \frac{5\pi}{8}\right) = 0$$

$$\therefore \quad x = \frac{15}{2\left(3 + \frac{5\pi}{8}\right)}$$

Since $y = 5 - x\left(1 + \frac{\pi}{4}\right)$,

when $x = \dfrac{15}{6 + \frac{5\pi}{4}}$

$$y = \frac{12}{5\pi + 24} + 2$$

To evaluate this on CAS, simply type in:

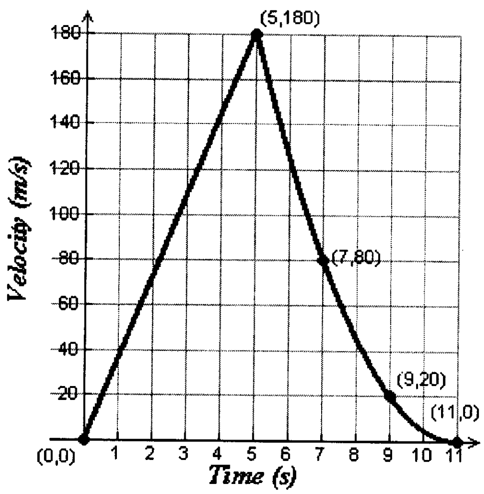
5 - x*(1+pi/4)|x=15/(6+5pi/4)

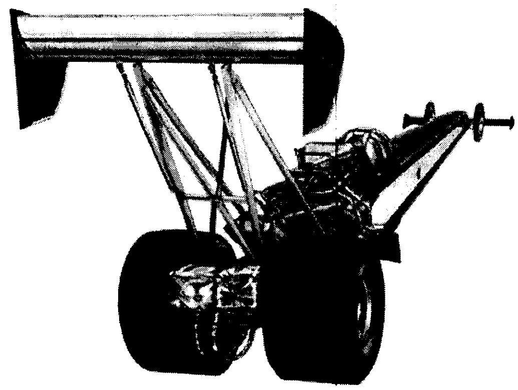
Section D (continued)

For Marker Use Only

Question 36

The motion of a dragster is modelled by the **velocity - time** graph below. The dragster accelerates at a constant rate for 5 seconds before deploying a parachute so as to decelerate. It comes to rest after 11 seconds. (6 marks)





(a) Determine the linear equation for the velocity, v (m/s), over the first 5 seconds and state the constant acceleration during this period.

$m = \frac{180-0}{5-0} = 36$

$V = 36t, 0 \leq t \leq 5$

$\text{Acceleration} = 36 \ m/s^2$

(b) Determine the distance covered before the parachute was deployed.

$x = \int 36t \, dt = 18t^2 + c$

When $t=0, x=0 \therefore c=0$

$x = 18t^2$ At $t=5, x = 18 \times 5^2 = 450 m$

(c) Determine a quadratic equation for the velocity whilst decelerating and calculate the total distance travelled by the dragster under parachute before coming to rest.

Let $v(t) = at^2 + bt + c$ $V(7) = 80$, $V(9) = 20$,

$V(11) = 0$ Solving: $a = 5, b = -110, c = 605$

$V = 5t^2 - 110t + 605, 5 \leq t \leq 11$

$x = \int 5t^2 - 110t + 605 \, dt$

$= \frac{5t^3}{3} - 55t^2 + 605t + c$

From previous part:

When $t=5, x = 450 \therefore c = \frac{-4225}{3}$

$\therefore x(t) = \frac{5t^3}{3} - 55t^2 + 605t - \frac{4225}{3}$ $\therefore x(11) = 810$

$\therefore$ Total distance $= 810 m$

For Marker Use Only

Section C

Answer **ALL** questions in this section.

This section assesses **Criterion 5**.

Question 29 (2 marks)

Find the ~~x~~ coordinates of the points on the curve $y = x^3 - 2x^2 + 5$ where the tangents to the curve are parallel to the line $y = 7 - x$.

$\frac{dy}{dx} = -1 \quad \therefore 3x^2 - 4x = -1$

$3x^2 - 4x + 1 = 0$

$\therefore (3x-1)(x-1) = 0 \quad \therefore x = \frac{1}{3}, 1$

$f\left(\frac{1}{3}\right) = \frac{130}{27} \quad f(1) = 4 \quad \therefore (1,4)$ and $\left(\frac{1}{3}, \frac{130}{27}\right)$

Question 30 (4 marks)

A TV satellite company has 10 000 customers each paying \$25 per month. Research has shown the company is likely to gain 500 more customers for every \$1 the monthly fee drops.

(a) Show that the revenue per month is $R = -500x^2 + 2500x + 250000$, where x represents the drop in the monthly fee in dollars.

If monthly fee drops by x dollars, then company gains $500x$ customers, each paying $25-x$ dollars

There are now $10000 + 500x$ customers paying $25-x$

$\therefore R = (10000 + 500x)(25-x) = -500x^2 + 2500x + 250{,}000$

(b) Apply differential calculus techniques to determine the fee per month that will provide maximum revenue. There is no need to justify that it is a maximum.

$R'(x) = -1000x + 2500$

$R'(x) = 0$

$\therefore 2500 - 1000x = 0$

$\therefore x = 2.5$

$\therefore$ Fee must be reduced to \$25 − 2.50

$= \$22.50$

Section C continues opposite.

For Marker Use Only

Section C

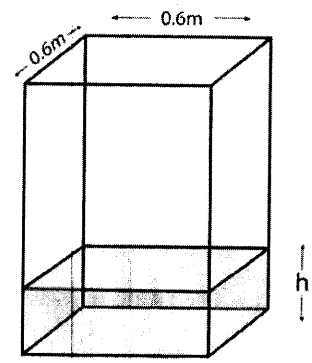
Answer **ALL** questions in this section.

This section assesses **Criterion 5**.

Question 29

A liquid is in a container which is a rectangular prism.
The depth of liquid, h (cm), in the tank at a time t minutes is given by the function with the rule: $h(t) = (4.5 - 0.3t)^3$.

(2 marks)



(a) Determine an expression for $h'(t)$.

$h'(t) = \dfrac{-81(t-15)^2}{1000}$

(b) Explain the meaning of $h'(4) = -9.801$.

At $t = 4$ minutes, the height of the liquid is decreasing at the rate of 9.801 cm/min.

Section C continues opposite.

For Marker Use Only

Section C

Answer **ALL** questions in this section.

This section assesses **Criterion 6**.

Question 29

(a) Determine algebraically the coordinates of the **three** points of $f(x) = x^4 - 4x^3 + 4x^2 - x$ which have a gradient of -1. (3 marks)

$f'(x) = -1$ $\quad f'(x) = 4x^3 - 12x^2 + 8x - 1$

$\therefore 4x^3 - 12x^2 + 8x - 1 = -1$

$\therefore 4x^3 - 12x^2 + 8x = 0$

$4x(x^2 - 3x + 2) = 0$

$4x(x-2)(x-1) = 0 \quad \therefore x = 0, 1, 2$

$f(0) = 0, \; f(1) = 0, \; f(2) = -2$

Points: $(0,0)$, $(1,0)$, $(2,-2)$

(b) By considering the tangents through these points, show that **two** of the points lie on the same tangent. (2 marks)

Tangent at $(0,0)$ is: $y = -x$

Tangent at $(1,0)$ is: $y = -x + 1$

Tangent at $(2,-2)$ is $y = -x$

$\therefore (0,0)$ and $(2,-2)$ lie on the same tangent

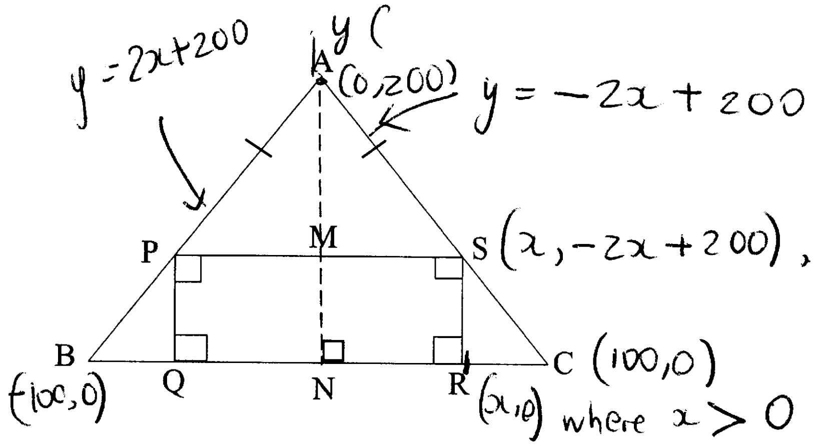
Use the Tangent line facility on CAS to get these equations.

(Use CALCULUS Tangentline$(f(x), x = a)$ to determine the equation of the tangent at $x = a$ on CAS)

Section C continues.

Question 5

An isosceles triangle ABC has a height of 200 mm and its base is 200 mm. Rectangle PQRS is an inscribed rectangle with P on AB and S on AC as shown in the diagram below.



a. Show that the area of PQRS is $400x - 4x^2$ if NR = x.

Area of PQRS $= 2x(200 - 2x)$

where $x > 0$

(since $\overline{QR} = 2x$ and $\overline{SR} = 200 - 2x$)

$\therefore A = 400x - 4x^2$

4 marks

b. Find the value for x that will maximise the area.

$A(x) = 400x - 4x^2$

$A'(x) = 400 - 8x$

Let $A'(x) = 0$ $\therefore x = 50$

2 marks

SECTION 2 – **Question 5** - continued

c. Find the maximum area of the rectangle PQRS.

$A(50) = 400 \times 50 - 4 \times 50^2$

$= 10,000$

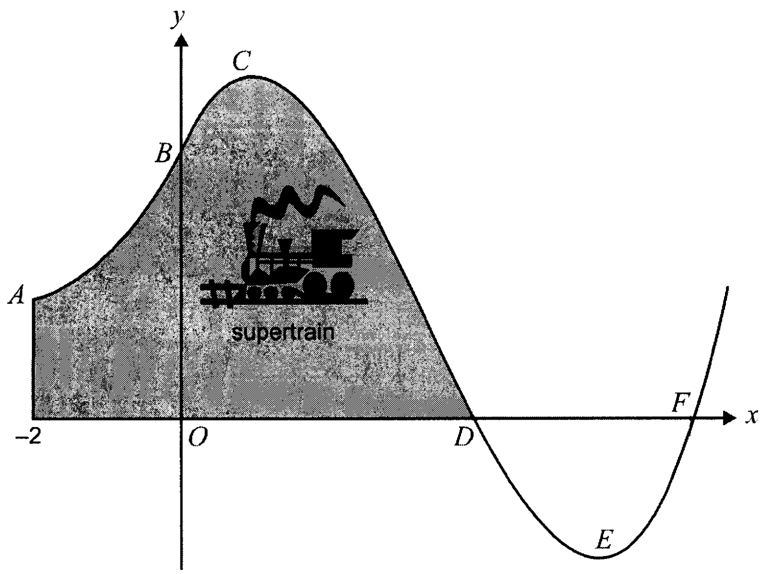
$\therefore$ Maximum area $= 10,000 \text{ mm}^2$

1 mark

Total 7 marks

END OF QUESTION AND ANSWER BOOK

Question 4



A part of the track for Tim's model train follows the curve passing through A, B, C, D, E and F shown above. Tim has designed it by putting axes on the drawing as shown. The track is made up of two curves, one to the left of the y-axis and the other to the right.

B is the point (0, 7).

The curve from B to F is part of the graph of $f(x) = px^3 + qx^2 + rx + s$ where p, q, r and s are constants and $f'(0) = 4.25$.

a. i. Show that $s = 7$.

$f(0) = 7 \quad \therefore 7 = p(0)^3 + q(0)^2 + r(0) + s$

$\therefore s = 7$

ii. Show that $r = 4.25$.

$f'(x) = 3px^2 + 2qx + r$

$f'(0) = 4.25$

$\therefore 4.25 = r$

1 + 1 = 2 marks

The furthest point reached by the track in the positive y direction occurs when $x = 1$. Assume $p > 0$.

b. **i.** Use this information to find q in terms of p.

$f'(1) = 0 \therefore 3p + 2q + 4.25 = 0$

$\therefore 2q = -4.25 - 3p$

$q = -2.125 - \frac{3p}{2}$

ii. Find $f(1)$ in terms of p.

Use CAS to assist!

Defining $f(x) = px^3 + qx^2 + 4.25 + 7$

and $q = \frac{-4.25 - 3p}{2}$

we find: $f(1) = \frac{73}{8} - \frac{p}{2}$

iii. Find the value of a in terms of p for which $f'(a) = 0$ where $a > 1$.

Solving $f'(x) = 0$ gives

$x = \frac{17}{12p}$ or $x = 1$

Since $a > 1$, $a = \frac{17}{12p}$

iv. If $a = \frac{17}{3}$, show that $p = 0.25$ and $q = -2.5$.

$\frac{17}{3} = \frac{17}{12p}$

$\therefore 12p = 3$

$\therefore p = \frac{1}{4}$ and

$q = \frac{-4.25 - 3 \times 0.25}{2}$

$= -2.5$

2 + 1 + 1 + 2 = 6 marks

For the following assume $f(x) = 0.25x^3 - 2.5x^2 + 4.25x + 7$.

c. Find the exact coordinates of D and F.

$f(x) = 0.25x^3 - 2.5x^2 + 4.25x + 7$

At D, F, $f(x) = 0$

$x = -1, 4, 7$

$\therefore D = (4, 0), \quad F = (7, 0)$

2 marks

d. Find the greatest distance that the track is from the x-axis, when it is below the x-axis, correct to two decimal places.

Let $f'(x) = 0$

$x = 1, \frac{17}{3}$

At $t, x = \frac{17}{3}$

$f\left(\frac{17}{3}\right) = \frac{-109}{27}$

1 mark

$\therefore$ Greatest distance below x-axis $= 3.70$ km

Question 4

Consider the function $f: R \rightarrow R, f(x) = (2-x)^2(3-x)^3$.

a. Find the x-coordinates of each of the stationary points of f and state the nature of each of these stationary points.

$f(x) = (2-x)^2(3-x)^3$

Let $f'(x) = 0$

$\therefore x = 3, 2, \frac{12}{5}$

$f'(x) = -(x-3)^2(x-2)(5x-12)$

x	0	2	2.1	$\frac{12}{5}$	2.8	3	4
$f'(x)$	−216	0	0.1215	0	−0.06	0	−16
Shape	\	—	/	—	\	—	\

At $x=2$: local minimum
At $x=\frac{12}{5}$, local maximum
At $x = 3$, stationary point of inflection

4 marks

Let $g: R \rightarrow R, g(x) = (a-x)^2(b-x)^3$, where a and b are real constants.

b. For what values of a and b will g have only one stationary point? State the nature of this stationary point.

Let $g(x) = (a-x)^2(b-x)^3$

$g'(x) = -(x-b)^2(x-a)(5x-3a-2b)$

For stationary points: $g'(x) = 0$

$\therefore x = a, b, \frac{3a+2b}{5}$

These will all be the same value if $b = a$.

2 marks

Now suppose that $b > a$.

c. Write down, in terms of a and b, the possible values of x for which $(x, g(x))$ is a stationary point of g.

$x = a, b, \frac{3a+2b}{5}$

2 marks

d. Let $h : R \to R$, $h(x) = (a-x)^m (b-x)^n$, where a and b are real constants, with $b > a$ and m and n are positive integers.

i. Write down in terms of a, b, m and n, the possible values of x for which $(x, h(x))$ is a stationary point of h.

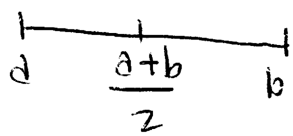
$x = a,\ b,\ \dfrac{an+bm}{m+n}$

Define h(x) on CAS and $dh(x) = \dfrac{d}{dx}(h(x))$

ii. For what values of m and n will h have a stationary point which is equidistant from $x = a$ and $x = b$?

3 + 1 = 4 marks

Total 12 marks



We require:

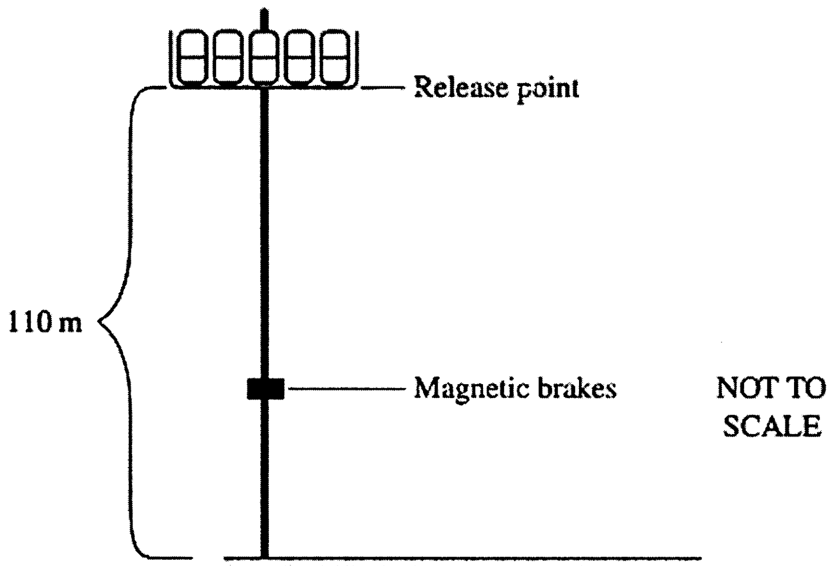
$\dfrac{a+b}{2} = \dfrac{an+bm}{m+n}$

$\therefore\ m = n.$

END OF QUESTION AND ANSWER BOOKLET

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) In a theme park ride, a chair is released from a height of 110 metres and falls vertically. Magnetic brakes are applied when the velocity of the chair reaches −37 metres per second.



The height of the chair at time t seconds is x metres. The acceleration of the chair is given by $a = -10$. At the release point, $t = 0$, $x = 110$ and $v = 0$.

(i) Using calculus, show that $x = -5t^2 + 110$. **2**

(ii) How far has the chair fallen when the magnetic brakes are applied? **2**

(i) $a = -10$

$\therefore v = \int -10\,dt = -10t + c$

When $t = 0$, $v = 0$ $\therefore c = 0$

$\therefore v = -10t$

$x = \int -10t\,dt = -5t^2 + c$

When $t = 0$, $x = 110$ $\therefore c = 110$ $\therefore x = -5t^2 + 110$

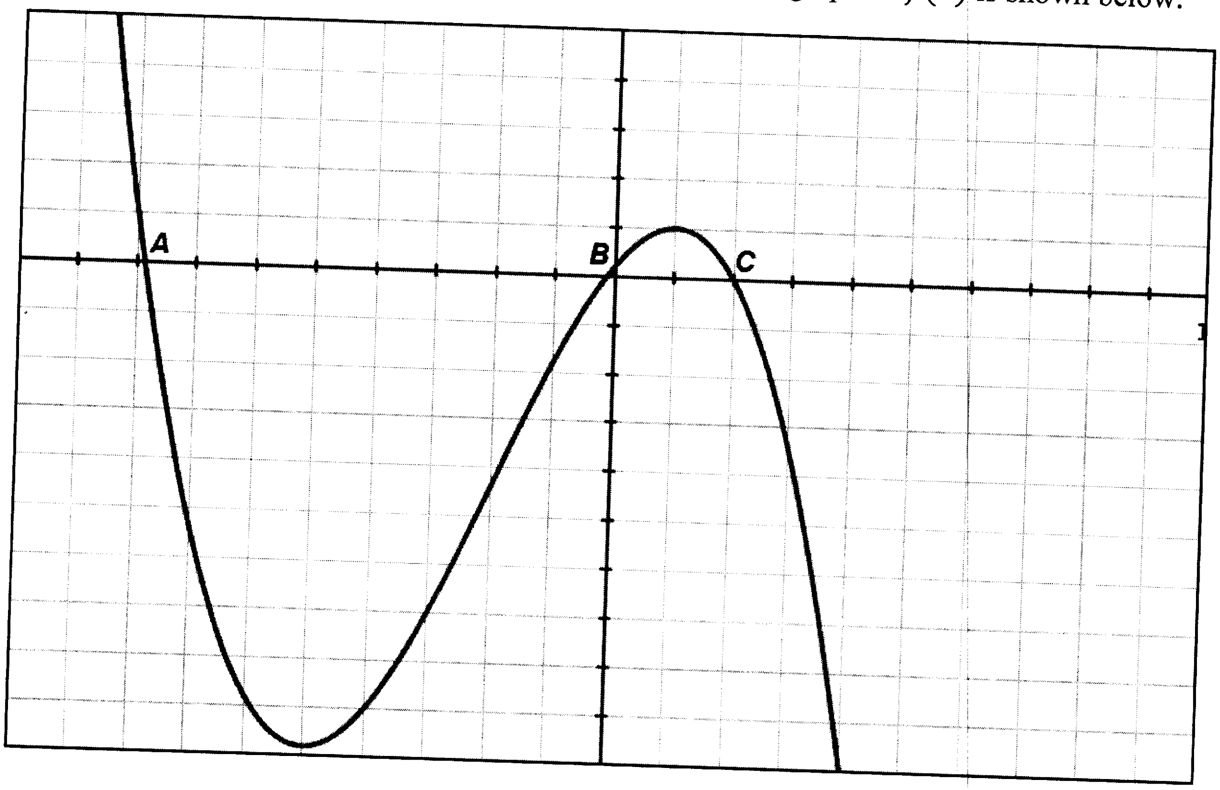
(ii) Brakes applied when $v = -37$ $\therefore -37 = -10t$, $t = 3.7$

When $t = 3.7$, $x = 110 - 5 \times (3.7)^2 = 41.55$ m

$\therefore$ It has fallen $110 - 41.55 = 68.45$ m

Question 5

Consider the function: $f(x) = 2 + 15x - 6x^2 - x^3$. The graph of $f(x)$ is shown below:



a. Determine the co-ordinates of the x-intercepts A, B and C (to two decimal places where appropriate).

$f(x) = 0$

$\therefore x = -7.87, \quad -0.13, \quad 2$

$A = (-7.87, 0), \quad B = (-0.13, 0), \quad C = (2, 0)$

3 marks

b. Use calculus to establish the co-ordinates of the stationary points.

$f'(x) = 15 - 12x - 3x^2$

Let $f'(x) = 0$

$\therefore 0 = 3x^2 + 12x - 15$

$0 = x^2 + 4x - 5$

$0 = (x+5)(x-1)$

$\therefore x = -5, 1$

$f(-5) = -98$

$f(1) = 10$

$\therefore (-5, -98)$

and

$(1, 10)$

4 marks

c. Find the equation of the tangent to the curve at the point where $x = -4$.

$f'(-4) = 15, \quad f(-4) = -90$

$\therefore (x_1, y_1) = (-4, -90), \; m = 15$

$y + 90 = 15(x + 4)$

$y = 15x - 30$

3 marks allocated so need to show working

3 marks

d. The tangent intersects the curve again. Find the co-ordinates of this point of intersection.

Let $f(x) = 15x - 30$

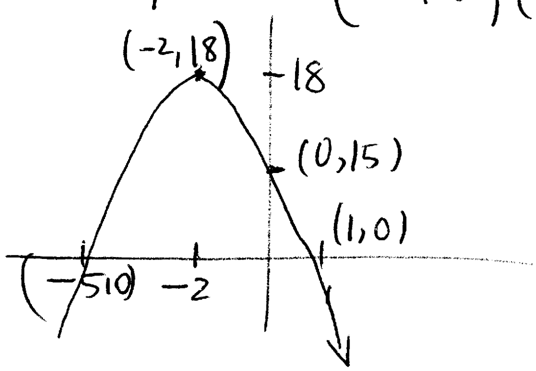
Solving: $x = -4, 2$

Second point is $(2, f(2)) = (2, 0)$

2 marks

e. Sketch the graph of the derivative function, $f'(x)$.

$f'(x) = -3(x+5)(x-1)$



2 marks

f. Hence, find the co-ordinates of the point of the steepest positive gradient on the graph of $y = f(x)$.

$f(x)$ has its maximum possible positive gradient when $x = -2$

$f(-2) = -44$

$\therefore$ Required point: $(-2, -44)$

2 marks

(Point where gradient is steepest on **original** graph required.)