QUESTION

Scientists are observing the behavior of a species of predator in an ecosystem. The population of the predator at time t months from the start of July is modeled by the equation:

$$N = 5000 - 1500\cos(\frac{\pi t}{6}).$$

a. What is the initial population?

$N(0) = 5000 - 1500\cos(0) = 3500$

b. i. What is the smallest value of the population?

3500

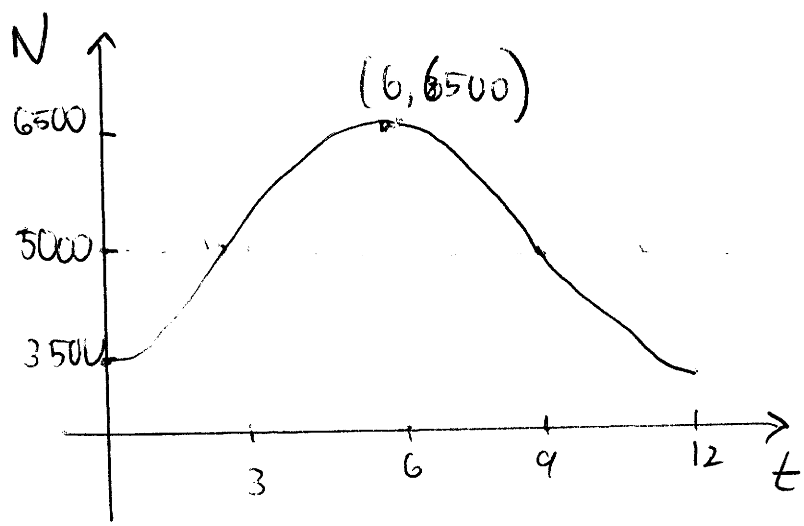
ii. What is the largest value of the population?

6500

c. What is the period of the function *N*?

$\frac{2\pi}{\frac{\pi}{6}} = 12$ months

d. Sketch the graph of *N* versus *t* for $0 \le t \le 12$.



e. Find the values of t within the domain $0 \le t \le 12$ for which the population is equal to 5750.

$$5750 = 5000 - 1500\cos\left(\frac{\pi t}{6}\right)$$

$$\frac{-1}{2} = \cos\left(\frac{\pi t}{6}\right) \quad \therefore \frac{\pi t}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\Rightarrow t = 4, 8$$

f. i. Find correct to three decimal places the values of t, where $0 \le t \le 12$, for which the population is equal to 6000.

$$6000 = 5000 - 1500\cos\left(\frac{\pi t}{6}\right)$$

$$t = 4.394, \quad 7.606$$

ii. find correct to two decimal places the number of months during the first twelve months for which the population exceeds 6000.

$$7.6063 - 4.3937$$

$$\simeq 3.21 \text{ months}$$

g. Calculate the average rate of change of the predator population from $t = 0$ to $t = 10$.

$$\frac{N(10) - N(0)}{10 - 0} = 75 \text{ predators/month}$$

Question 4 (11 marks)

The temperature inside a greenhouse is controlled by an air conditioning system in such a way that the temperature, T °C, at time t hours after 3:00 am is given by the rule $T(t) = 22 - 4\cos\left(\frac{\pi t}{12}\right)$.

a. Find the amplitude and period of function T. 2 marks

Period $= \dfrac{2\pi}{\frac{\pi}{12}} = 24$

Amplitude $= 4$

b. Find the maximum and minimum temperatures. 2 marks

$\boxed{T_{max} = 26 \quad T_{min} = 18}$

$+4$, -4: 26, 22, 18

c. **i.** Find $T(8)$ and state the time of day when this temperature occurs. 2 marks

$T(8) = 22 - 4\cos\left(\frac{2\pi}{3}\right)$

$= 22 - 4 \times -\frac{1}{2}$

$= 24$ at 11:00 AM

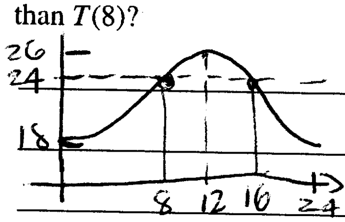
ii. When is the next time that the temperature in the greenhouse is $T(8)$? 1 mark

$24 = 22 - 4\cos\left(\frac{\pi t}{12}\right)$

$\therefore -\frac{1}{2} = \cos\left(\frac{\pi t}{12}\right)$ $\quad \frac{\pi t}{12} = \frac{2\pi}{3}, \frac{4\pi}{3}$ $\quad \therefore t = 8, 16$

Next time: 7:00 PM

iii. Over one day, for what fraction of time is the temperature in the greenhouse less than $T(8)$? 2 marks



$T < 24$ for $8 + 8 = 16$ hours

Fraction required: $\frac{16}{24} = \frac{2}{3}$

d. The maximum temperature in the greenhouse on any day is reduced to 25°C, whilst the minimum temperature remains the same. The temperature in the greenhouse, T °C, at time t hours after 3:00 am is now given by the rule $T(t) = b - a\cos\left(\frac{\pi t}{12}\right)$.

Determine the values of a and b. 2 marks

b: 25, 21.5, 18

$b = \frac{25 + 18}{2} = 21.5$

$a = 25 - 21.5 = 3.5$

$\boxed{a = 3.5 \quad b = 21.5}$

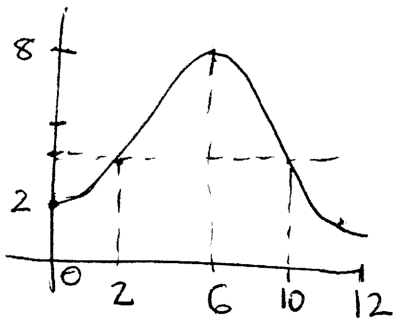
Question

The depth D of the water at the entrance to a harbour t hours after low tide can be modelled by the function: $D = a + b\cos(ct)$ for the constants a, b and c.

At $t = 0$, it is low tide and the depth is 2 metres.

At the next high tide, 6 hours later, the depth is 8 metres.

a. Give exact values for a, b and c.



Period = 12

$\therefore \frac{2\pi}{c} = 12 \quad \therefore c = \frac{\pi}{6}$

+3 ↑ 8

a = 5

−3 ↓ 2

$a = \frac{8+2}{2} = 5$

$b = -3$

$$\boxed{a = 5,\ b = -3,\ c = \frac{\pi}{6}}$$

b. Hence write down and solve a trigonometric equation to find how soon after high tide a vessel requiring a depth of at least 3.5 metres of water will not be able to enter the harbour.

$$5 - 3\cos\left(\frac{\pi t}{6}\right) = 3.5, \quad 0 \leq t \leq 12$$

Solving : $t = 2, 10$

$\therefore$ $10 - 6 = 4$ hours after high tide, the ship will not be able to enter.

For Marker Use Only

Section B

Answer **ALL** questions in this section.

This section assesses **Criterion 4**.

Question 25

(2 marks)

The temperature on a particular day can be modelled by the function $C = -6\cos\left(\frac{\pi}{12}t\right) + 20$, where t is the time elapsed in hours after 5:00 am and C is the temperature in degrees celsius. Calculate the temperature predicted by the model at 7:00 am on the same morning.

Express your answer correct to 1 decimal place.

At 7:00 AM, $t = 2$

$C = -6\cos\left(\frac{\pi}{6}\right) + 20$

$= 14.8°$

Question 26

→ Third Quadrant

(4 marks)

If $\pi < x < \frac{3\pi}{2}$ and $\cos x = -\frac{2}{5}$,

(a) Determine the exact value for $\sin x$.

$(\cos x)^2 + (\sin x)^2 = 1$

$\therefore \left(-\frac{2}{5}\right)^2 + \sin^2 x = 1$

$(\sin x)^2 = 1 - \frac{4}{25} = \frac{21}{25}$

$\therefore \sin x = \pm\frac{\sqrt{21}}{5}$ Since x is in 3rd Q, $\sin x < 0$ $\therefore \sin x = \frac{-\sqrt{21}}{5}$

(b) Hence, find the exact value for $\tan x$.

$\tan(x) = \frac{\sin x}{\cos x} = \frac{\sqrt{21}}{2}$

Section B continues opposite.

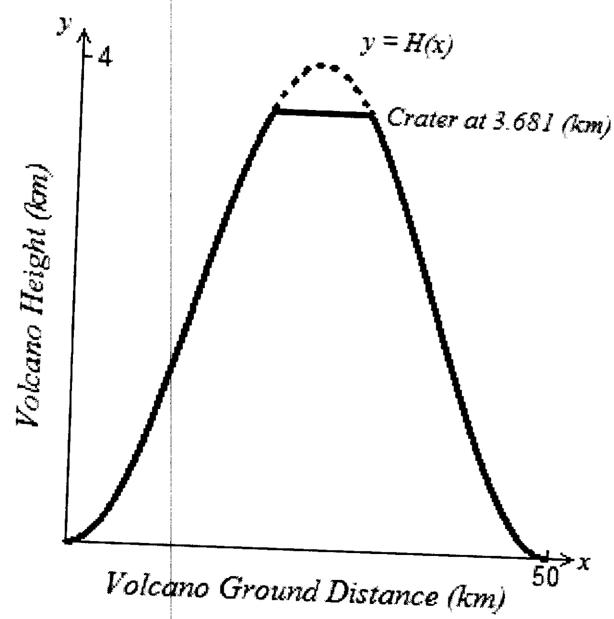
Section B (continued)

For Marker Use Only

Question 27

(4 marks)

The profile of a volcano is shown in the graph below.



A cosine function of the form $H(x) = a\cos(nx) + b$ can model the profile before a crater was formed as a result of an eruption.

(a) Determine exact values for a, n and b, given the maximum height before the eruption was 4 km whilst its base spans 50 km over 1 complete period.

$\frac{2\pi}{n} = 50 \quad \therefore n = \frac{\pi}{25}$

$+2 \uparrow$ 4; $-2 \downarrow$ 2, 0

$b = 2, \ a = -2$

$H(x) = -2\cos\left(\frac{\pi x}{25}\right) + 2$

(b) Determine the distance across the crater formed after the eruption, given it is parallel to the ground and at a height of 3.681 km.

Express your answer to the nearest km.

$3.681 = -2\cos\left(\frac{\pi x}{25}\right) + 2, \ 0 \leq x \leq 50$

Solving: $x = 20.4435, \ 29.5565$

$\therefore$ Distance $= 9.113$ km

$\simeq 9$ km

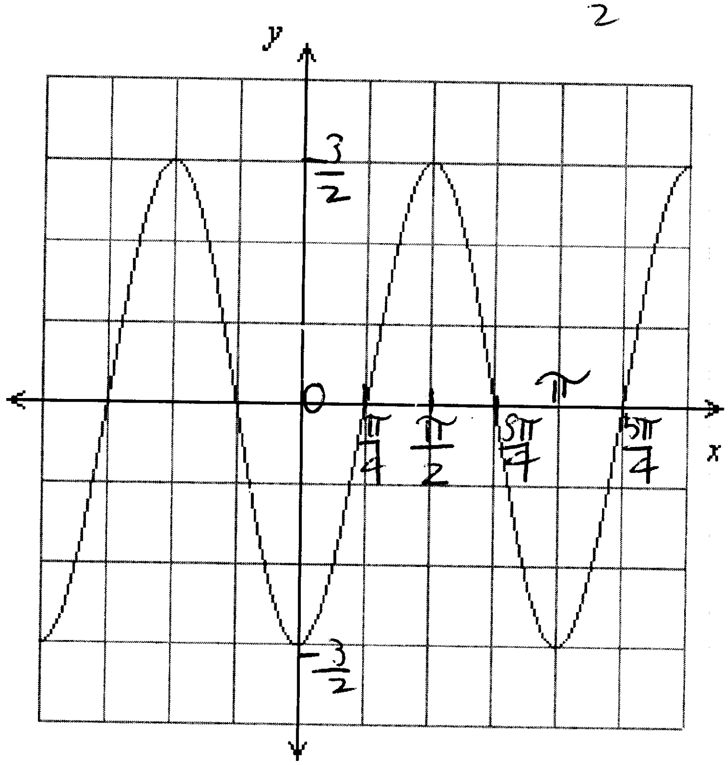
Section B continues over the page.

For Marker Use Only

Question 10

Put appropriate scales on both the x and y axes given the graph below is of the trigonometric function $y = -\frac{3}{2}\cos(2x)$. (2 marks)

Period $= \frac{2\pi}{2} = \pi$



Question 11

Express 210° in radians:

(a) As an exact value (1 mark)

$210^\circ = 210 \times \frac{\pi}{180}^c$

$= \frac{7\pi}{6}^c$

(b) As a decimal value (*correct to three decimal places*) (1 mark)

3.665^c

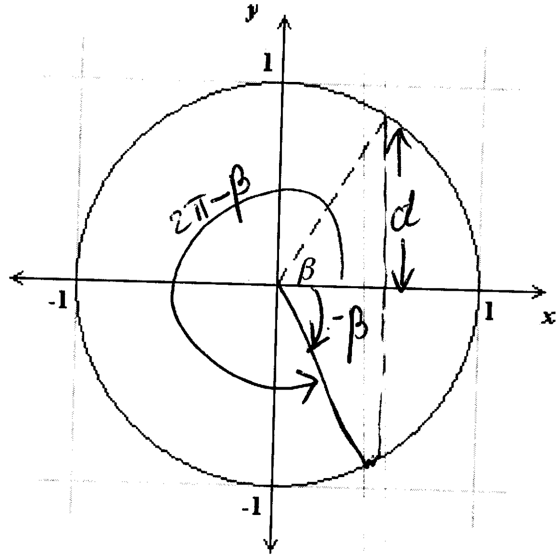
Question 16

For Marker Use Only

The unit circle below has an acute angle of β drawn in the first quadrant. (4 marks)

(a) If the value of $\sin(\beta) = d$, **label** on the diagram opposite:

- length d
- acute angle $(-\beta)$
- reflex angle $(2\pi - \beta)$



(b) Hence, determine an expression for $\tan(2\pi - \beta)$ in terms of d.

$\tan(2\pi - \beta) = -\tan(\beta)$

$\sin(\beta) = d$

so $\cos(\beta) = \sqrt{1-d^2}$

$\therefore \tan(\beta) = \dfrac{d}{\sqrt{1-d^2}}$

$\therefore \tan(2\pi-\beta) = -\dfrac{d}{\sqrt{1-d^2}}$

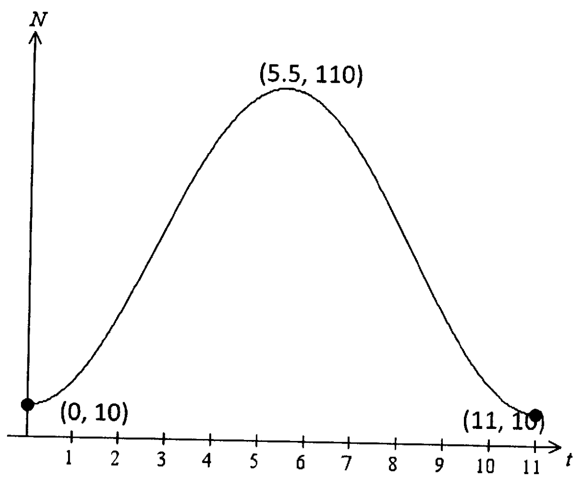
Question 2

Solar flares or "sunspots" are caused by the Sun's magnetic field. The average number of sunspots in any given year follows a periodic cycle, called a solar cycle.

Using historical data, Bryan, a solar astronomer, modelled the number of sunspots during a solar cycle with the function

$N:[0,11]\to R,\ \ N(t)=b-a\cos(nt),$

where N is the number of sunspots t years after the start of a solar cycle and a, b and n are **positive** real constants. The graph of the function is shown.



Period = 11 years

a. According to this model:

i. How many complete solar cycles have occurred between 1755 and 2008?

2008 − 1755 = 253

253 ÷ 11 = 23 ∴ 23 Complete cycles

ii. What is the range of N?

[10, 110]

1 + 1 = 2 marks

b. Show that $n=\dfrac{2\pi}{11}$

$\dfrac{2\pi}{n}=11$

$\therefore n=\dfrac{2\pi}{11}$

1 mark

c. Show that $a = 50$ and $b = 60$.

$+50 \uparrow$ $b + a = 110$

$b = 60$

$-50 \downarrow$ $b - a = 10$

$$\boxed{\begin{aligned} b &= \frac{110+10}{2} = 60 \\ a &= 110 - 60 = 50 \end{aligned}}$$

2 marks

Assume that a new solar cycle began on 1 January 2009.

d. What is the predicted number of sunspots on 1 January 2011, correct to the nearest integer?

$N(t) = 60 - 50\cos\left(\frac{2\pi t}{11}\right)$

When $t = 2$, $N(2) = 60 - 50\cos\left(\frac{4\pi}{11}\right) \simeq 39$

1 mark

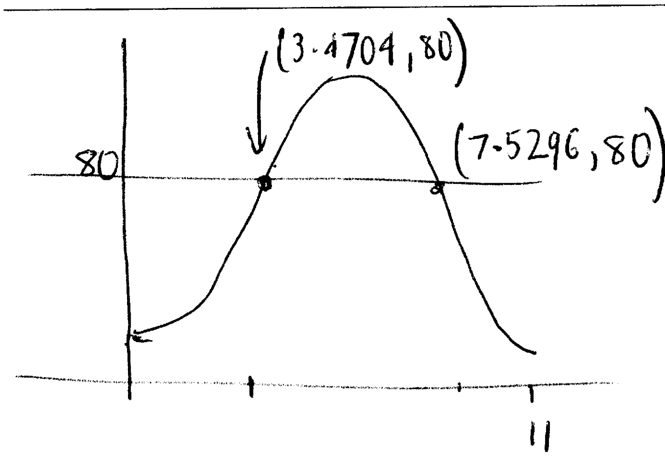
e. The level of UV radiation increases with the number of sunspots. Bryan proposes to monitor UV radiation levels during the period when $N \geq 80$. For what length of time is $N \geq 80$? Express the answer correct to the nearest month.

Required time $= 7.5296 - 3.4704$ years

$= 4.0592$ years

$= 48.7$ months

$\simeq 49$ months



2 marks

Section B (continued)

For Marker Use Only

Question 27

Let $f(x) = p\cos(4x) + q$, where p and q are constants and $p > 0$. (4 marks)

(a) State the maximum and minimum values of $f(x)$, in terms of p and q.

$q + p$

$+p$ q

$q - p$

$f_{max} = q + p$

$f_{min} = q - p$

(b) Under what conditions for p and q, will $f(x) \le 0$, for all values of x?

$q + p \le 0$

$q \le -p$

(c) Under what conditions for p and q, will $f(x) > 2$, for all values of x?

$q - p > 2$

$q > p + 2$

Section B continues over the page.

Section B (continued)

For Marker Use Only

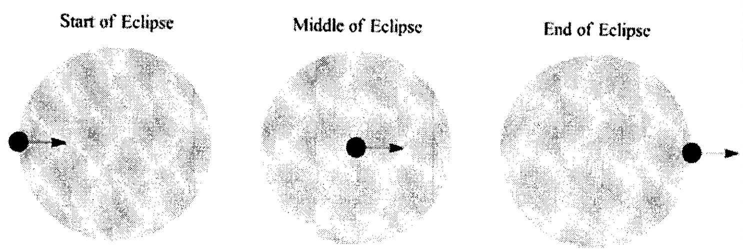
Question 30

(3 marks)

The planet Kepler 296f passes between the star Kepler 296 and Earth every 63 days and 8 hours, obscuring the light. This is known as an eclipse.

During the eclipse, the observed brightness, B, of the star varies according to the formula $B = a\cos(nt) + c$, where a, n and c are constants. The time, t (in hours), is measured from when the planet starts to pass in front of the star.

The observed brightness of the star decreases from 100% when the eclipse starts to a minimum of 99.898% when the planet is directly in the middle of the star and the brightness returns to 100% once the eclipse ends. The duration of the eclipse is 3 hours and 45 minutes.



Determine the values of a, n and c.

0.051 ↑ 100
c — 99.949
0.051 ↓ 99.898

$C = (100 + 99.898) \div 2$
$= 99.949$

Period = 3.75 hours

$\therefore \frac{2\pi}{n} = 3.75$ $\quad\therefore \frac{2\pi}{n} = \frac{15}{4}$

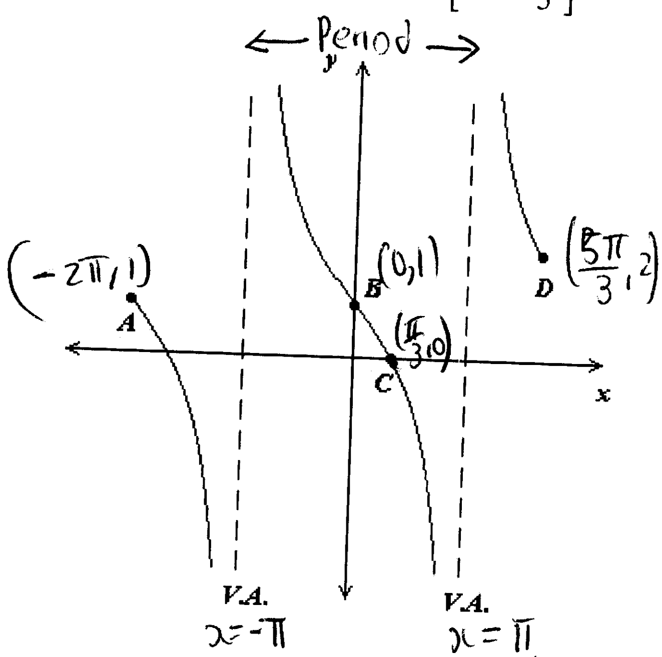
$\therefore n = \frac{8\pi}{15}$

$\boxed{\therefore a = 0.051, \; C = 99.949, \; n = \frac{8\pi}{15}}$

For Marker Use Only

Question 17

The graph below is of the form $y = a\tan(bx) + c$ for x: $\left[-2\pi, \frac{5\pi}{3}\right]$.



The labelled points have coordinates: $A\ (-2\pi, 1)$, $B\ (0, 1)$, $C\left(\frac{\pi}{3}, 0\right)$ and $D\left(\frac{5\pi}{3}, 2\right)$.

The vertical asymptotes shown are at $x = -\pi$ and $x = \pi$.

Find possible values for a, b and c, and state a possible equation for the function. (6 marks)

Period $= 2\pi$ $\therefore$ $2\pi = \frac{\pi}{b}$ $\therefore b = \frac{1}{2}$

$y = a\tan\left(\frac{x}{2}\right) + c$

When $x = 0$, $y = 1$ $\therefore$ $c = 1$

$y = a\tan\left(\frac{x}{2}\right) + 1$

When $x = \frac{\pi}{3}$, $y = 0$

$\therefore 0 = a\tan\left(\frac{\pi}{6}\right) + 1$

$-1 = a\tan\left(\frac{\pi}{6}\right)$

$-1 = \frac{a}{\sqrt{3}}$ $\therefore a = -\sqrt{3}$

$y = -\sqrt{3}\tan\left(\frac{x}{2}\right) + 1$

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (7 marks)

The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2013.

a. Find the period and amplitude of the function n. 2 marks

$$\text{Period} = \frac{2\pi}{\frac{\pi}{3}} = 6 \text{ months}$$

$$\text{Amplitude} = 400$$

b. Find the maximum and minimum populations of wombats in this location. 2 marks

1600

1200

800

$$n_{max} = 1600$$

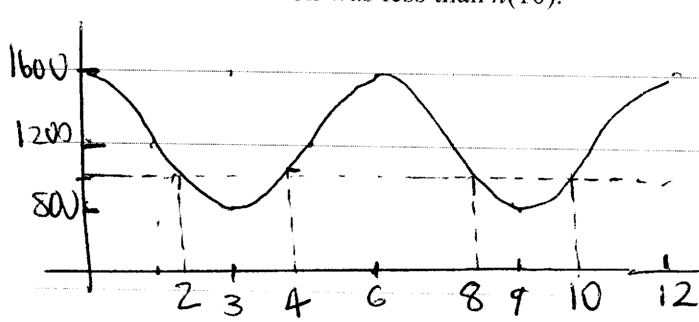
$$n_{min} = 800$$

c. Find $n(10)$. 1 mark

$$n(10) = 1200 + 400\cos\left(\frac{10\pi}{3}\right)$$

$$= 1200 - 400 \times \frac{1}{2} = 1000$$

d. Over the 12 months from 1 March 2013, find the fraction of time when the population of wombats in this location was less than $n(10)$. 2 marks



Solving $n(t) = 1000$ for $0 \leq t \leq 12$ (or using the intersection command on CAS)

Total time $n < 1000 = (4-2) + (10-8) = 4$ months

$\therefore$ Fraction $= \frac{4}{12} = \frac{1}{3}$

Question 2

A proposal for the temperature ranges at which foods can be safely kept is shown in the table below.

Zone	Temperature range in degrees Celsius
Cool safe zone	less than 5
Unsafe zone	5 - 60
Hot safe zone	greater than 60

A particular food has a temperature over a 24-hour period that can be described by the function

$$T(t) = 30\sin\left(\frac{\pi t}{6}\right) + 30, \qquad 0 \le t \le 24$$

where T is the temperature in degrees Celsius, and t is the time in hours where $t = 0$ corresponds to midnight on Thursday.

a. What is the temperature of this particular food at midnight on Thursday?

$T(0) = 30\sin(0) + 30 = 30$

$\therefore 30^\circ$

1 mark

b. Explain whether or not the temperature of this food is ever in the "hot safe zone".

60
30
0

No, since $0 \le T \le 60$ for all values of t.

1 mark

c. State the time(s) when this particular food is at $60^0 C$.

$\sin\left(\frac{\pi t}{6}\right) = 1 \qquad 0 \le t \le 24$

$0 \le \frac{\pi t}{6} \le 4\pi$

1 mark

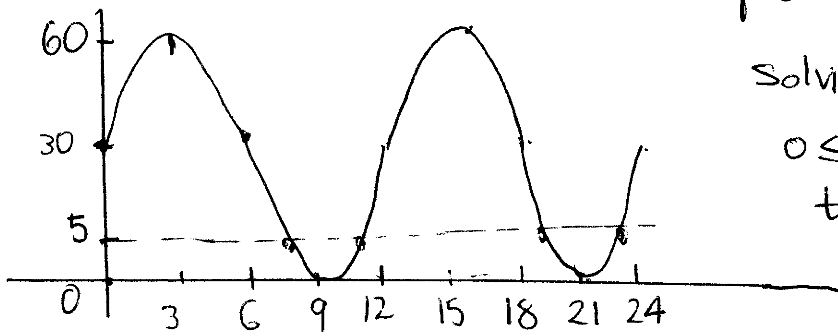
$\frac{\pi t}{6} = \frac{\pi}{2}, \frac{5\pi}{2} \quad \therefore t = 3, 15$

$\therefore$ 3:00 AM and 3:00 PM

$$\text{Period} = \frac{2\pi}{\frac{\pi}{6}} = 12$$

d. Between which times is this food safe? Express times to the nearest minute where appropriate.

When $T \leq 5$, the food is safe.



Solving $T(t) = 5$ for $0 \leq t \leq 24$ gives $t = 7.881, 10.119, 19.881, 22.119$

Food is safe between 7:53 am and 10:14 am and also between 7:53 pm and 10:14 pm

e. At what times on Friday is the rate at which the temperature of this food is changing negative?

Function is decreasing (negative gradient) for $3 < t < 9 \;\cup\; 15 < t < 21$

$\therefore$ Temperature is decreasing between 3:00 am and 9:00 am, and also between 3:00pm and 9:00pm.

Section B (continued)

For Marker Use Only

Question 28 (6 marks)

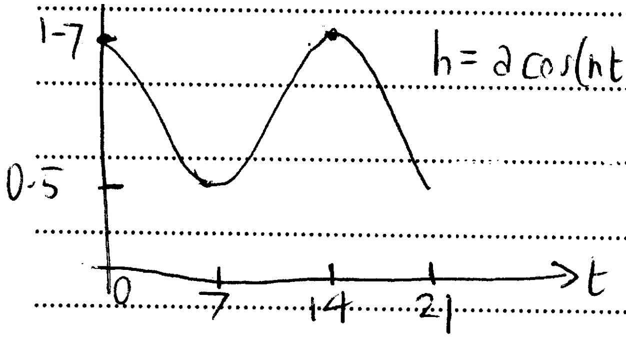
A boat left Hobart at high tide at 2:00 pm on Tuesday. The high tide at this time was measured at 1.7 m.

The boat returned to Hobart at low tide at 11:00 am on Wednesday. The low tide at this time am was measured at 0.5 m.

While the boat was out to sea there was one low tide and one high tide.

Note: Low tide is the time when the local water level of the ocean is at its lowest. High tide is the time when the local water level of the ocean is at its highest.

(a) Determine an equation that models the tidal height as a function of time (in hours) since 2:00pm on Tuesday



$h = a\cos(nt) + c$

$t = 0 \Rightarrow$ 2:00pm Tuesday

$t = 21 \Rightarrow$ 11am Wednesday

Period = 14

$\therefore \frac{2\pi}{n} = 14 \quad \therefore n = \cancel{7}\ \frac{\pi}{7}$

$0.6\uparrow \left[\begin{array}{l} 1.7 \\ 1.1 \\ 0.5 \end{array}\right. 0.6\downarrow$

$h = 0.6\cos\left(\frac{\pi t}{7}\right) + 1.1$

(b) Use your model to determine at what time(s) of day on **Wednesday** the tidal height is exactly 0.8 m. Give the time(s) in hours and minutes. $10 \le t \le 34$

$0.8 = 0.6\cos\left(\frac{\pi t}{7}\right) + 1.1$

$-\frac{1}{2} = 0.6\cos\left(\frac{\pi t}{7}\right) \quad 10 \le t \le 34$

Solving for t gives: $t = \frac{56}{3}, \frac{70}{3}, \frac{98}{3}$

(t = hours after 2:00 pm Tuesday) $t = 18\frac{2}{3}, 23\frac{1}{3}, 32\frac{2}{3}$

Times: 8:40am, 1:20pm, 10:40 pm on Wednesday

NOTE: Drawing the graph on CAS makes it clear: