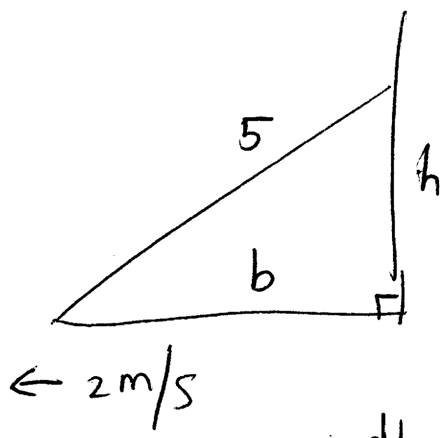
EXTENDED RESPONSE.

Q1.



Variables : b, h, t

$$\frac{db}{dt} = 2$$

Find : $\frac{dh}{dt}$

$$\frac{dh}{dt} = \frac{dh}{db} \cdot \frac{db}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{dh}{db} \times 2$$

$$b^2 + h^2 = 5^2$$

$$\therefore h^2 = 25 - b^2$$

$$h = \sqrt{25 - b^2}$$

$$\therefore h = (25 - b^2)^{\frac{1}{2}}$$

$$\frac{dh}{db} = \frac{1}{2}(25 - b^2)^{-\frac{1}{2}} \times 2b$$

$$= \frac{-b}{\sqrt{25 - b^2}}$$

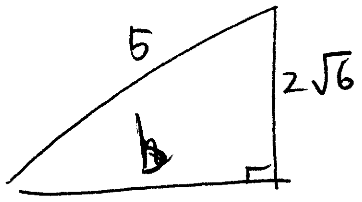
$$\therefore \frac{dh}{dt} = 2 \times \frac{-b}{\sqrt{25 - b^2}}$$

When $b = 3$, $\frac{dh}{dt} = \frac{-2 \times 3}{\sqrt{25 - 9}} = \frac{-6}{\sqrt{16}} = -\frac{3}{2}$ m/s

$\therefore$ Sliding down at $\frac{3}{2}$ m/s

(ii)

Initial position:



$5^2 = (2\sqrt{6})^2 + b^2$

$25 = 24 + b^2$

$\therefore b^2 = 1$

$b = 1$

$\therefore$ Initially, $b = 1$.

At time t, $b = 1 + 2t$ (since the value of b increases at 2m/s)

The ladder will be lying flat when $b = 5$

$\therefore \quad 1 + 2t = 5$

$\therefore t = 2$

(b) (i) $\dfrac{dA}{dt} = 2\log_e(t+1)$

$\therefore A = \int 2\log_e(t+1)\,dt$

$A = 2(t+1)\log_e(t+1) - 2t + c$

At $t=0$, $A=0$

$\therefore \quad 0 = 2\log_e 1 - 0 + c \quad \therefore c = 0$

$\therefore \quad A = 2(t+1)\log_e(t+1) - 2t$

(ii) When $t = 30$

$A = 2(31)\log_e(31) - 60 \simeq 152\cdot 9 \text{ m}^2$

(iii) Let required value of t be T:

$$A(T) - A(30) = 50$$

$$\therefore A(T) = A(30) + 50$$

Define: $a(t) = 2(t+1)\log_e(t+1) - 2t$

Solve $a(T) = a(30) + 50$

gives $T = 37.0619$

$\therefore$ It takes him a further 7 minutes to paint another $50m^2$.

Q2.

(a) $a = 15$

(ii) $c = 15$

(iii) Period $= 1000$

$$\therefore \frac{2\pi}{\frac{\pi}{b}} = 1000$$

$$\therefore 2b = 1000$$

$$b = 500.$$

(b) $y = 15 + 15\sin\left(\frac{\pi x}{500}\right)$

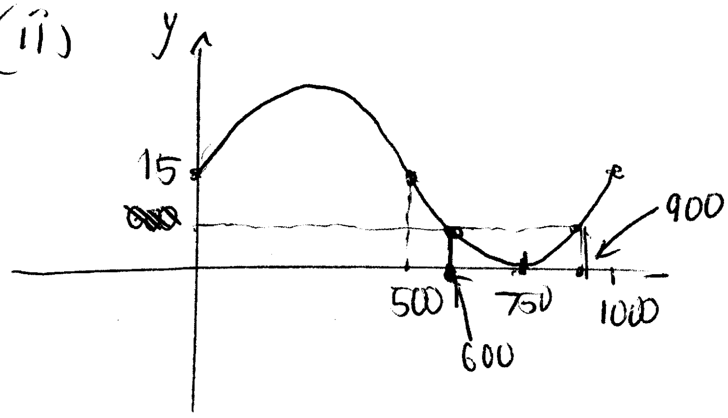
(c) (i) When $x = 600$, $y = 15 + 15\sin\left(\frac{600\pi}{500}\right)$

$\therefore y = 15 + 15\sin\left(\frac{6\pi}{5}\right)$

$y \simeq 6.18322$ m

$y \approx 618$ cm

(ii)

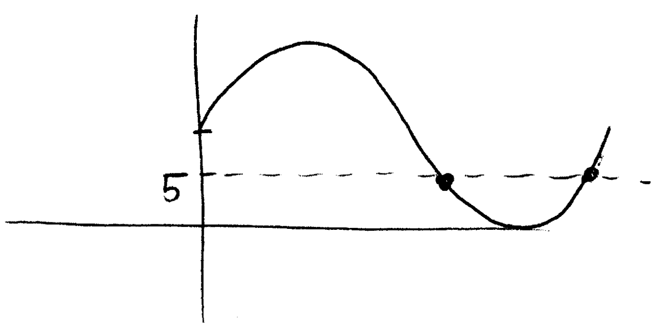


By symmetry

$y(600) = y(900)$

$\therefore$ At $x = 900$ depth is the same.

(d) (i)



Solving $15 + 15\sin\left(\frac{\pi x}{500}\right) = 5$, $0 \leq x \leq 1000$

gives: $x = 616.14$ or $x = 883.86$

$\therefore$ Water will be suitable for ~~616 ≤ x ≤ 884~~

$0 \leq x \leq 616$ or $884 \leq x \leq 1000$

$\therefore x \in [0, 616] \cup [884, 1000]$

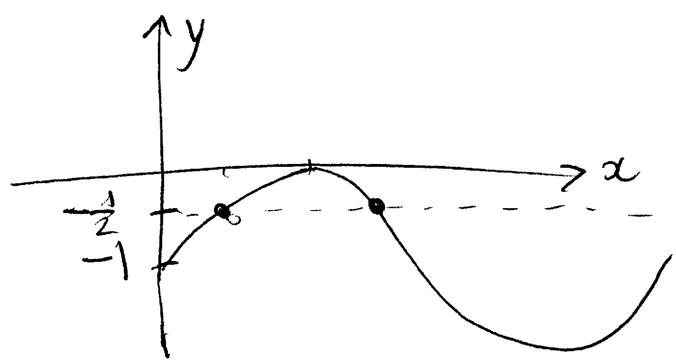
(d) (ii) Length of race

$= 616.14 - 0$

≈ 616 m.

(e) $\left(\frac{\pi}{2}, 0\right)$

(f)



Solve : $\sin x - 1 = -\frac{1}{2}$ for $0 \le x \le 2\pi$

$\therefore \sin x = \frac{1}{2}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore$ Tourists can walk where $x \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

(g)

When $x = 750$, $g(750) = -10\sin\left(\frac{750\pi}{500}\right) - 10$

$= -10\sin\left(\frac{3\pi}{2}\right) - 10 = 0.$

When $x = 875$, $g(875) = -10\sin\left(\frac{875\pi}{500}\right) - 10$

$= -10\sin\left(\frac{7\pi}{4}\right) - 10$

$= \frac{10\sqrt{2}}{2} - 10 = 5\sqrt{2} - 10.$

$\therefore$ The two end points of slide are:

$(750, 0)$ and $(875, 5\sqrt{2}-10)$

$$\therefore m = \frac{5\sqrt{2}-10-0}{875-750}$$

$$= \frac{5\sqrt{2}-10}{125}$$

$$= \frac{\sqrt{2}-2}{25}$$

$\therefore$ Equation:

$$y - 0 = \frac{\sqrt{2}-2}{25}(x-750)$$

$$y = \frac{\sqrt{2}-2}{25}(x-750)$$

(h)

$$y = 15 + 15\sin\left(\frac{\pi x}{500}\right) \text{ m}$$

Let $f(x) = 15 + 15\sin\left(\frac{\pi x}{500}\right)$

$$-f(x) = -15 - 15\sin\left(\frac{\pi x}{500}\right)$$

$$-\frac{2}{3}f(x) = -10 - 10\sin\left(\frac{\pi x}{500}\right)$$

$$\therefore \quad g(x) = -\frac{2}{3}f(x)$$

$\therefore$ Required transformations:

1. Dilate by factor $\frac{2}{3}$ away from x-axis
2. Reflect in x-axis.

Q3.

(a) (i) Since all events are independent,

$$\Pr(R \cap S) = \Pr(R) \cdot \Pr(S)$$

$$\therefore \Pr(R \cap S) = 0.25 \times 0.75$$

$$= 0.1875$$

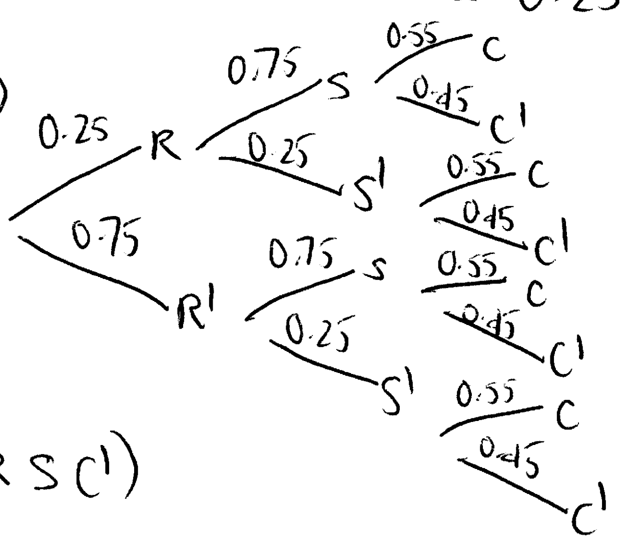
(ii) $$\Pr(R \mid S) = \frac{\Pr(R \cap S)}{\Pr(S)}$$

$$= \frac{\Pr(R)\Pr(S)}{\Pr(S)}$$

$$= \Pr(R)$$

$$= 0.25$$

(b)

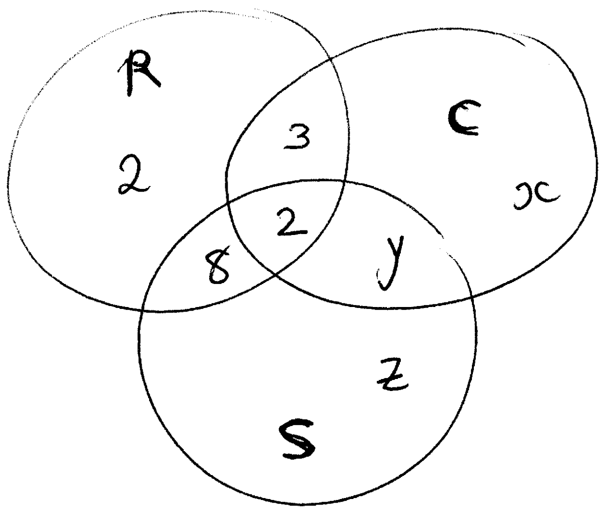


$\Pr(R\,S\,C')$

$$= 0.25 \times 0.75 \times 0.45 = \frac{1}{4} \times \frac{3}{4} \times \frac{9}{20}$$

$$= \frac{27}{320}$$

(b)(iii)



$Pr(R) = 0.25$

$\therefore n(R) = 15$

$Pr(C) = 0.55$

$\therefore n(C) = 33$

$\therefore \quad 5 + y + x = 33$

$\therefore y + x = 28$

$n(S) = 45$ since $Pr(S) = 0.75$

$\therefore \quad 10 + y + z = 45$

$y + z = 35$ (2)

$15 + y + x + z = 60$

$y + x + z = 45$ (3)

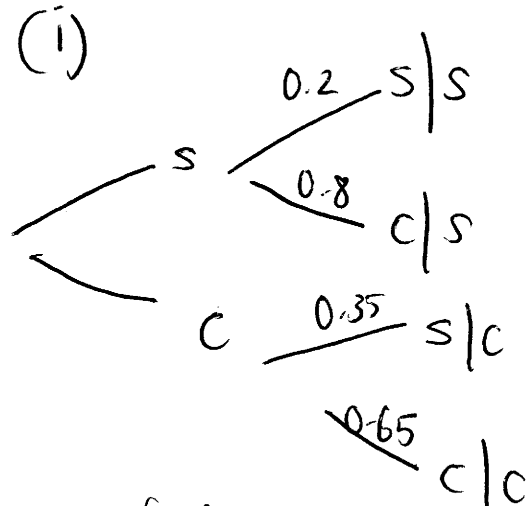
Since $y + x = 28 \Rightarrow z = 17$.

Since $y + z = 35 \Rightarrow y = 18$.

Since $x + y = 28 \Rightarrow x = 10$

$\therefore$ 10 staff members take C only.

(c) (i)



$$T = \begin{bmatrix} 0.2 & 0.35 \\ 0.8 & 0.65 \end{bmatrix} \qquad S_1 = \begin{array}{c} \begin{bmatrix} 45 \\ 15 \end{bmatrix} \end{array}\begin{array}{l} S \\ C \end{array}$$

$$S_2 = \begin{bmatrix} 0.2 & 0.35 \\ 0.8 & 0.65 \end{bmatrix}\begin{bmatrix} 45 \\ 15 \end{bmatrix} = \begin{bmatrix} 0.2\times45 + 0.35\times15 \\ 0.8\times45 + 0.65\times15 \end{bmatrix}\begin{array}{l} S \\ C \end{array}$$

$\therefore$ Number taking cycle $= 0.8\times45 + 0.65\times15$

$= 45.75 \approx 46$

(ii) $S_5 = T^4 S_1$

$$= \begin{bmatrix} 0.2 & 0.35 \\ 0.8 & 0.65 \end{bmatrix}^4 \begin{bmatrix} 45 \\ 15 \end{bmatrix} = \begin{bmatrix} 18.3 \\ 41.7 \end{bmatrix}\begin{array}{l} S \\ C \end{array}$$

$\therefore$ 18 will take Step class on Friday.

(iii) $S_{steady} = \begin{bmatrix} \frac{0.35}{1.15} \\ \frac{0.8}{1.15} \end{bmatrix} = \begin{bmatrix} \frac{35}{115} \\ \frac{80}{115} \end{bmatrix}\begin{array}{l} S \\ C \end{array} = \begin{bmatrix} \frac{7}{23} \\ \frac{16}{23} \end{bmatrix}$

$\therefore$ Long term no. doing step $= \frac{7}{23} \times 60 \simeq 18$

and 42 doing Cycle

Q3 (d)

$\underline{0.6}\ S\ \underline{0.6}\ S\ \underline{0.2}\ C\ \underline{0.2}\ C\ \underline{0.2}\ C$

$(0.6)^2 \times (0.2)^3 = 0.00288$

(e) Let X = no. who take Step

$X \stackrel{d}{=} Bi(n=10, p=0.6)$

$Pr(X>6) = Pr(X=7) + Pr(X=8) + Pr(X=9) + Pr(X=10)$

$\simeq 0.3823$

(f) Let Y = no of ~~science staff~~ cycle classes taken

$Pr(Y \geq 2) > 0.65$

Solve : $\therefore 1 - Pr(Y=0) - Pr(Y=1) = 0.65$

$0.35 = Pr(Y=0) + Pr(Y=1)$

$\therefore 0.35 = \binom{n}{0}(0.3)^0(0.7)^n + \binom{n}{1} 0.3 \times (0.7)^{n-1}$

$\therefore 0.35 = (0.7)^n + n \times 0.3 \times (0.7)^{n-1}$

Solving : $n = 6.755$

$\therefore n = 7$

Q4.

(a) $f(x) = -x^4 + 8x^3 - 24x^2 + 32x - 15$

Let $f(x) = A(x-B)^4 + C$

~~Expanding: f(x) =~~ Clearly, $A = -1$ (since coefficient of x^4 is -1)

$\therefore f(x) = -(x-B)^4 + C$

$\therefore f(x) = -x^4 + 4bx^3 - 6b^2x^2 + 4b^3x - b^4 + C$

$\therefore 4b = 8 \quad \therefore b = 2$

$C - b^4 = -15$

$\therefore C - 16 = -15 \quad \therefore C = 1$

$\therefore f(x) = -(x-2)^4 + 1$

(b) $f(1) = f(b)$

$f(1) = -(1-2)^4 + 1 = -(-1)^4 + 1 = 0$

Clearly, $f(3) = 0$ also since

$f(3) = -(3-2)^4 + 1 = 0$

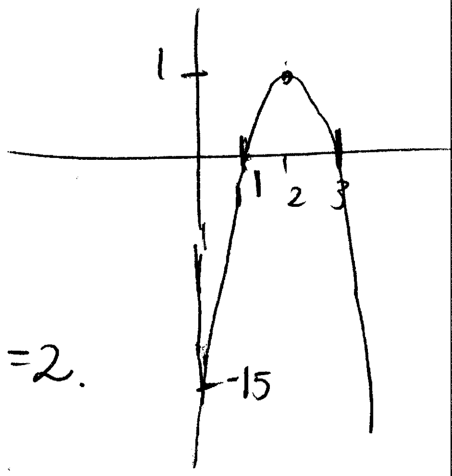
$\therefore b = 3.$

(c) (i) $[1, 3]$

(ii) $(1, 3)$

(d)

Sketch: $f(x) = -(x-2)^4 + 1$



In the interval
$x \in [1, 3]$
~~When~~ there is a stationary point at $x = 2$.
$\therefore c = 2$.

(e) $g\left(\frac{5}{2}\right) = \left|\log_e\left(\frac{1}{2}\right)\right|$

$g\left(\frac{7}{2}\right) = \left|\log_e\left(\frac{3}{2}\right)\right|$

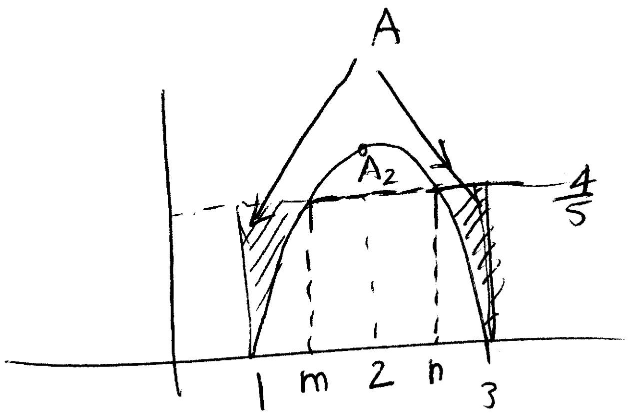
$\therefore g\left(\frac{5}{2}\right) \neq g\left(\frac{7}{2}\right)$ so Rolle's Theorem does not apply.

(i) $k = \frac{1}{3-1}\int_1^3 -(x-2)^4+1 \, dx$

$= \frac{1}{2}\int_1^3 -(x-2)^4+1 \, dx$

$= \frac{4}{5}$

(ii)



By the property of average value of a function, the area below "missing" from the rectangle must equal the area above it.

By symmetry, the two shaded areas are equal

By definition of the average value of a function:

$A_2 = A + A$

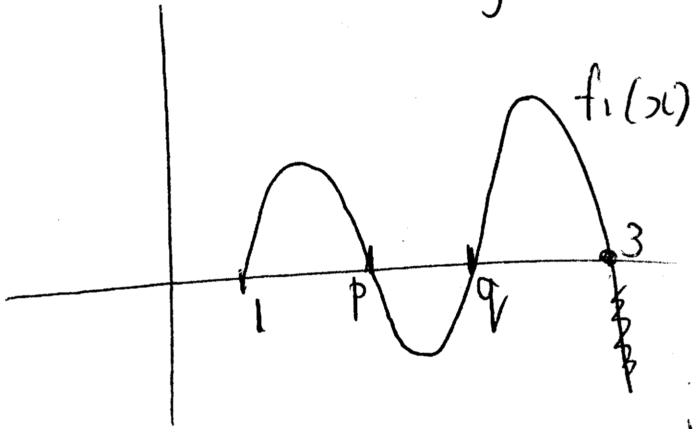
$\therefore 2\int_1^m \left(\frac{4}{5} - f(x)\right) dx = \int_m^n \left(f(x) - \frac{4}{5}\right) dx$

To find the values of m and n we need to solve: $\frac{4}{5} = -(x-2)^4 + 1$ where $x \in [1,3]$

Solving: $x = 1.3313,\ 2.6687$

$m = 1.3313,\ n = 2.6687$

(9) By Rolle's Theorem, if there are three stationary points in the interval $[1,3]$ and $f_1(1) = f_1(3) = 0$ then there must be four intercepts in the interval $[1,3]$ (including $x=1$ and $x=3$)



$f(p) = f(1) = 0$
$f(p) = f(q) = 0$
$f(q) = f(3) = 0$

Therefore, this function must have an equation of form

$$f_1(x) = -(x-1)(x-3)(x-p)(x-q)$$

Expanding:

$$f_1(x) = -x^4 + (p+q+4)x^3 - (pq + 4p + 4q + 3)x^2 + (4pq + 3p + 3q)x - 3pq$$

But according to the question,

$$f_1(x) = -x^4 + 8x^3 + ax^2 + bx + d$$

$\therefore \quad p + q + 4 = 8$

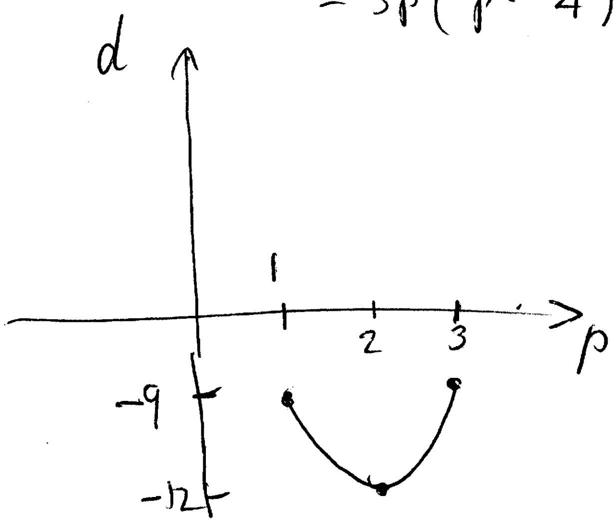
$\therefore \quad p + q = 4 \quad$ where $p \in [1,3]$
$q \in [1,3]$

$d = -3pq \quad$ But $q = 4 - p$

$\therefore \quad d = -3p(4-p)$ where $1 \leq p \leq 3$

We now consider the function:

$$d = -3p(4-p) \text{ for } 1 \leq p \leq 3$$
$$= 3p(p-4)$$



When $p=1$, $d = 3 \times -3$
$= -9$

When $p = 3$, $d = -9$

When $p=2$, $d = 6 \times -2$
$= -12$

Clearly, $-12 \leq d \leq -9$.

$\therefore$ For three stationary points between $x=1$ and $x=3$ the values of d where $f_1(x) = -x^4 + 8x^3 + ax^2 + bx + d$ are: $d \in [-12, -9]$