SOLUTIONS

PARADECOLLEGE

Mathematics Methods 3&4

SAC 2 2013

Topics:

- **Exponential and Logarithmic functions ;**
- **Function, relation and transformations;**
- **Circular Functions;**
- **Modeling and Inverse function and;**
- **Composition of functions**

9	**Short answer questions**	**54 marks**
1	**Analysis question**	**11 marks**
	Total Marks:	**65 marks**

Name of Student: /65 = %

(Which is 10% of the 34% allocated marks for the school based assessment tasks)

This Task is technology free.

Students are not allowed to bring any calculators.

Students are **permitted to bring into the room for this task: pens, pencils, highlighters, erasers, sharpeners and rulers, bound summary booklet**

No Calculators.

Question 1

For the function $f:(-1,\infty)\rightarrow R, f(x)=\frac{1}{3}\log_e\left(\frac{x+1}{2}\right)$

a. Find the rule for the inverse function, f^{-1}. 2 marks

$$y=\frac{1}{3}\log_e\left(\frac{x+1}{2}\right)$$

$$\downarrow$$

$$x=\frac{1}{3}\log_e\left(\frac{y+1}{2}\right)$$

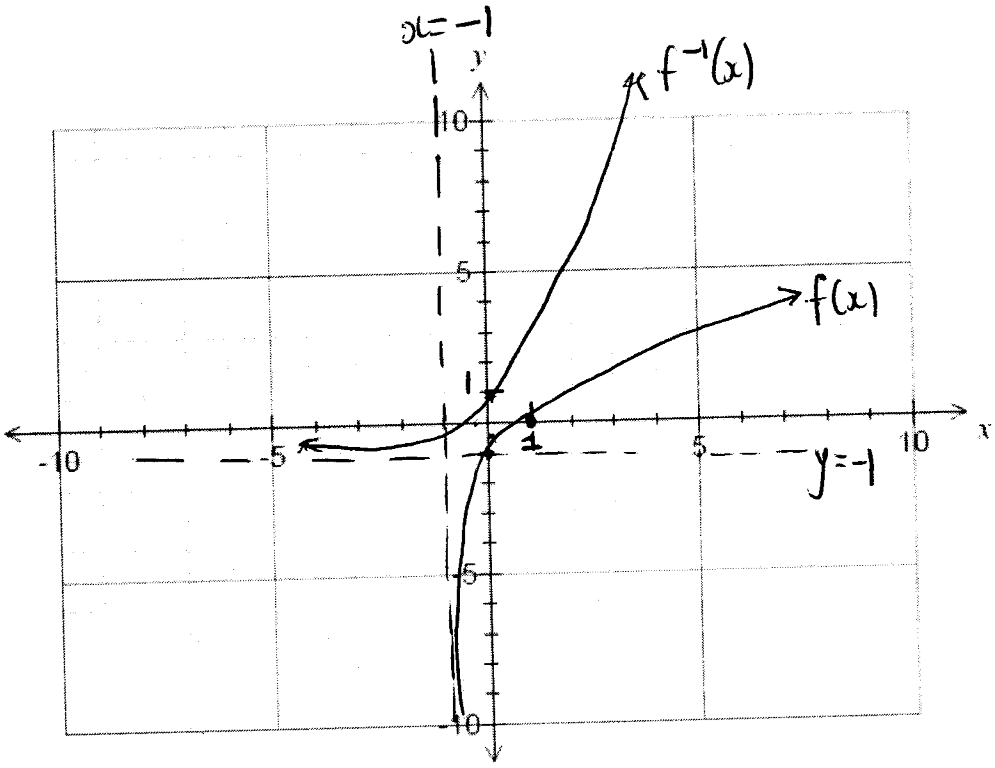
$$3x=\log_e\left(\frac{y+1}{2}\right)$$

$$e^{3x}=\frac{y+1}{2}$$

$$f^{-1}(x)=2e^{3x}-1$$

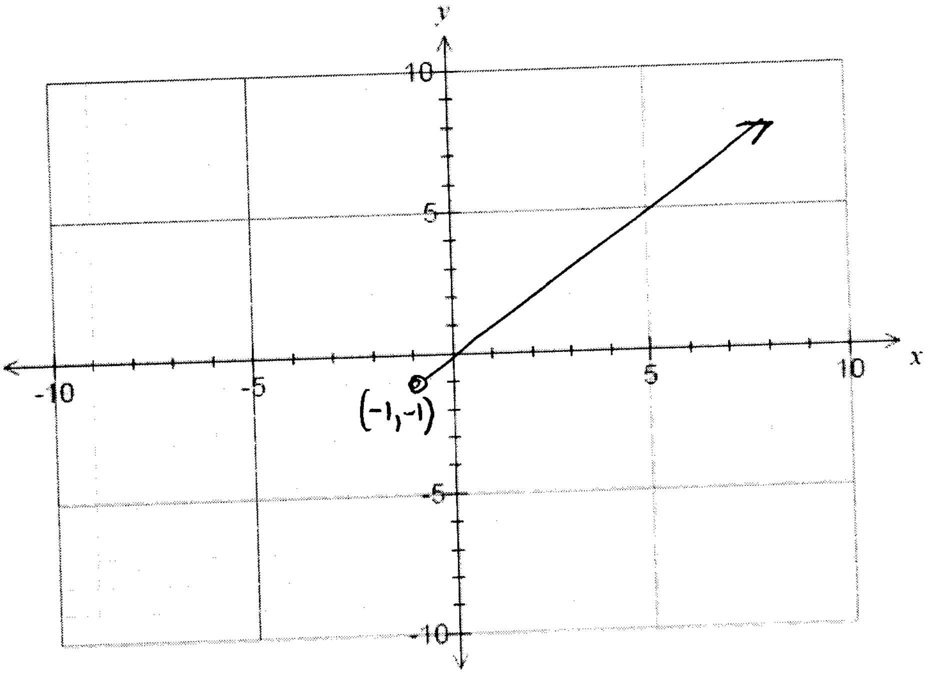
dom(f)	ran(f)
(-1,∞)	R
dom(f⁻¹)	ran(f⁻¹)
R	(-1,∞)

b. Sketch the graphs of $f(x)$, $f^{-1}(x)$ on the same set of axes below.. 2 marks



c. Find and then sketch below $f^{-1}(f(x))$. 2 marks

$f^{-1}(f(x)) = x, \quad x \in (-1, \infty)$



Total Question 1 = 6 marks

Question 2

Consider $f(x) = x^2 + 2x$ and $g(x) = e^x$

a. State the rule of $f(g(x))$, its domain and its range 3 marks

$$f(g(x)) = (e^x)^2 + 2e^x$$

$$= e^{2x} + 2e^x, \quad x \in R$$

Range: $(0,\infty)$

b. Solve $f(g(x)) = 3(e^x + 2)$ for x. Write the solution(s) in exact form. 3 marks

$$e^{2x} + 2e^x = 3e^x + 6$$

$$e^{2x} - e^x - 6 = 0$$

Let $e^x = p$

$$p^2 - p - 6 = 0$$

$$(p-3)(p+2) = 0$$

$$p = 3, -2$$

$\therefore e^x = 3$ or $e^x = -2$

$\downarrow$ No solution

$\therefore x = \log_e 3$

Total Question 2 = 6 marks

Question 3

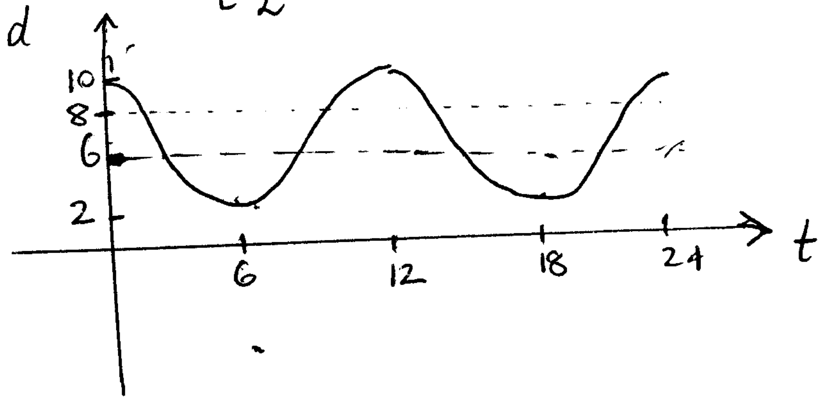
The depth *d* metres of water at the entrance to a harbour at time *t* hours from midnight on a particular day can be modelled by the equation. $d = 4\cos\frac{\pi t}{6} + 6, 0 \le t \le 24$.

a. Sketch a graph showing the times when the depth of the water is 6 metres?

3 marks

$$\text{Period} = \frac{2\pi}{\frac{\pi}{6}} = 12$$

Range: [- 10, - 6, - 2



b. Findwhat length of time over the 24 hour period from midnight is the depth of water 8 metres or more?

3 marks

Solving : $4\cos\left(\frac{\pi t}{6}\right) + 6 = 8$

$\cos\left(\frac{\pi t}{6}\right) = \frac{1}{2}$, $0 \le t \le 24$

$\therefore 0 \le \frac{\pi t}{6} \le 4\pi$

$\frac{\pi t}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$t = 2, 10, 14, 22$

Required length of time

$= (2-0) + (14-10) + (24-22)$

$= 8$ hours.

Total Question 3 = 6 marks

Question 4

The temperature T°C of water t minutes after being poured into a cup can be modelled by $T = T_0 e^{-kt}$, $t \geq 0$. The water is initially boiling (100°C) and after 10 minutes it has a temperature of 70°C.

a State the value of T_0. 1 mark

When $T=0$, $T = 100$

$\therefore 100 = T_0 \times e^0 \quad \therefore T_0 = 100$

b Show that the exact value of k is $\frac{1}{10}\log_e \frac{10}{7}$. 2 marks

$T = 100\, e^{-kt}$

When $t = 10$, $T = 70$

$70 = 100\, e^{-10k}$

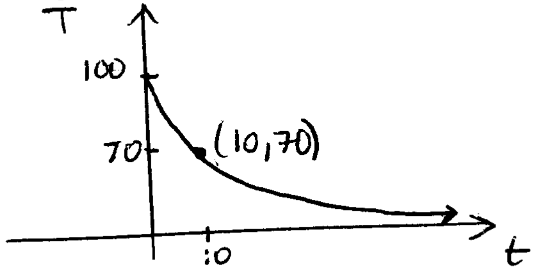
$\frac{70}{100} = e^{-10k}$

$\therefore \frac{100}{70} = e^{10k}$

$\rightarrow 10k = \log_e\left(\frac{10}{7}\right)$

$k = \frac{1}{10}\log_e\left(\frac{10}{7}\right)$

c Sketch the graph of T versus t. 1 marks



d. The model for the temperature can be expressed in the form $T = T_0 a^{t/10}$, $t \geq 0$. Find the exact value of a. 2 mark

$T = 100 \times e^{\frac{t}{10}\log_e\left(\frac{10}{7}\right)}$

$= 100 \times \left(e^{\log_e\left(\frac{10}{7}\right)}\right)^{\frac{t}{10}}$

$= 100 \times \left(\frac{10}{7}\right)^{t/10}$

$\therefore a = \frac{10}{7}$

Total Question 4 = 6 marks

Question 5

a. Describe the transformation required to transform the graph of $y = \sin(x)$ to the graph of $y = 4\sin 2\left(x - \frac{2\pi}{3}\right) + 5$.

2 marks

Dilate by a factor of 4 from x-axis

Dilate by a factor of $\frac{1}{2}$ from y-axis

Translate $\frac{2\pi}{3}$ units in positive x direction

Translate 5 units in positive y direction

b. Find the exact solutions of $4\sin 2\left(x - \frac{2\pi}{3}\right) + 5 = 7$ over the domain $[0, \pi]$.

4 marks

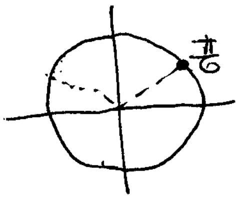
$$4\sin\left(2x - \frac{4\pi}{3}\right) + 5 = 7$$

$$0 \le x \le \pi$$

$$0 \le 2x \le 2\pi$$

$$-\frac{4\pi}{3} \le 2x - \frac{4\pi}{3} \le \frac{2\pi}{3}$$

$$\sin\left(2x - \frac{4\pi}{3}\right) = \frac{1}{2}$$



$$2x - \frac{4\pi}{3} = -\frac{7\pi}{6}, \frac{\pi}{6}$$

$$2x = \frac{4\pi}{3} - \frac{7\pi}{6}, \frac{4\pi}{3} + \frac{\pi}{6}$$

$$2x = \frac{8\pi}{6} - \frac{7\pi}{6}, \frac{8\pi}{6} + \frac{\pi}{6}$$

$$2x = \frac{\pi}{6}, \frac{3\pi}{2}$$

$$x = \frac{\pi}{12}, \frac{3\pi}{4}$$

Total Question 5 = 6 marks

Q 6.(a) $\log_e (x+1)^2 - \log_e(x^2-1) = \log_e(x-1)$

$\therefore \log_e \left(\frac{(x+1)^2}{x^2-1}\right) = \log_e(x-1)$

$\therefore \log_e \left(\frac{(x+1)^2}{(x-1)(x+1)}\right) = \log_e(x-1)$

$\frac{x+1}{x-1} = x-1$

$x+1 = (x-1)^2$

$x+1 = x^2-2x+1$

$0 = x^2-3x$

$x(x-3) = 0$

$x = 0, 3$

But $x=0$ is not a valid solution

$\therefore x = 3$.

(b) (i) Since there is a vertical asymptote at $x = \frac{3}{4}$

$\frac{3}{4} b + c = 0$ ①

Since $f(1) = 0$: $0 = a \log_e (b+c)$ $\therefore b+c = 1$ ②

① - ②: $-\frac{1}{4} b = -1$

$\therefore b = 4, c = -3$.

(ii) $f(x) = a \log_e (4x-3)$

Since $f\left(\frac{3}{2}\right) = \log_e 9$:

$\log_e 9 = a \log_e \left(4 \times \frac{3}{2} - 3\right)$

$\therefore \log_e 9 = a \log_e 3$

$\therefore 2 \log_e 3 = a \log_e 3$

$\therefore a = 2$.

Question 7

For $f: R \to R, f(x) = |x|$ and $g(x) = \sin(\pi x) - 3$:

a find:

i the rule for $f(g(x))$ 1 mark

$f(g(x)) = |\sin(\pi x) - 3|$

ii the rule for $g(f(x))$ 1 mark

$g(f(x)) = \sin(\pi |x|) - 3$

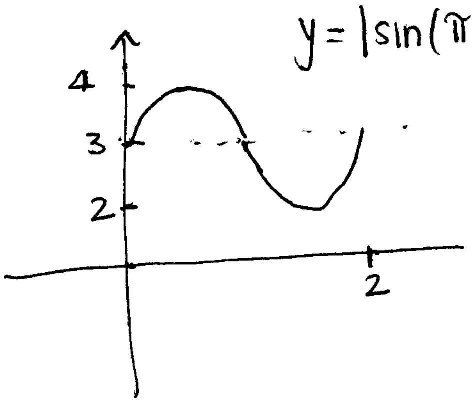
b state the range of each of the functions 2marks

Range of $f(g(x)) = [2, 4]$

Range of $g(f(x)) = [-4, -2]$

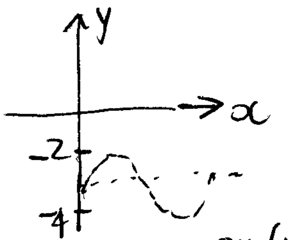
c sketch the graph of $y = f(g(x))$ 2 marks

$y = |\sin(\pi x) - 3|$

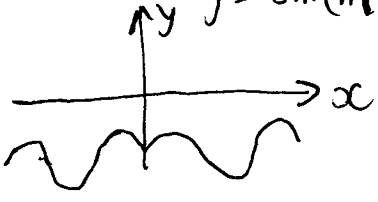


$y = \sin(\pi x) - 3$

-2
3
-4



$y = \sin(\pi |x|) - 3$



Total Question 7 = 6 marks

Question 8

For f: $(0, \infty) \rightarrow R, f(x) = \log_e(2x)$ and g: $R \rightarrow R$, $g(x) = e^{x+1}$:

a find $f(g(x))$ and $g(f(x))$ and express each in the form $mx + c$ where m and c are constants.

2 marks

$$f(g(x)) = \log_e(2e^{x+1})$$

$$= \log_e 2 + \log_e e^{x+1}$$

$$= x + 1 + \log_e 2$$

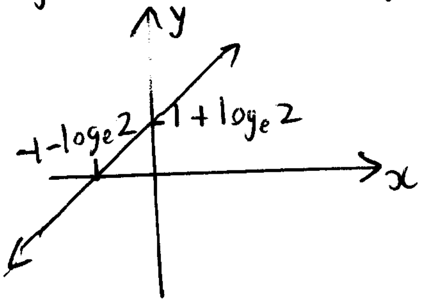
$$g(f(x)) = e^{\log_e 2x + 1} = e^1 . e^{\log_e 2x} = e . 2x$$

$$= 2ex$$

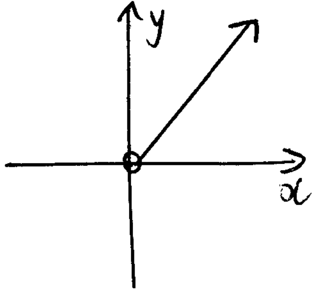
b sketch the graph of $y = f(g(x))$ and $y = g(f(x))$

2 marks

$f(g(x)) = x + 1 + \log_e 2$, $x \in \mathbb{R}$



$g(f(x)) = 2ex$, $x > 0$



c find the rule and domain of $g^{-1}(x)$

2marks

$y = e^{x+1}$

$\downarrow$

$x = e^{y+1}$

$y + 1 = \log_e x$

$\therefore g^{-1}(x) = \log_e x - 1$, ~~$x \in \mathbb{R}$~~ $x > 0$

dom(g)	ran(g)
$\mathbb{R}$	$(0, \infty)$
dom(g^{-1})	ran(g^{-1})
$(0, \infty)$	$\mathbb{R}$

Total Question 8 = 6 marks

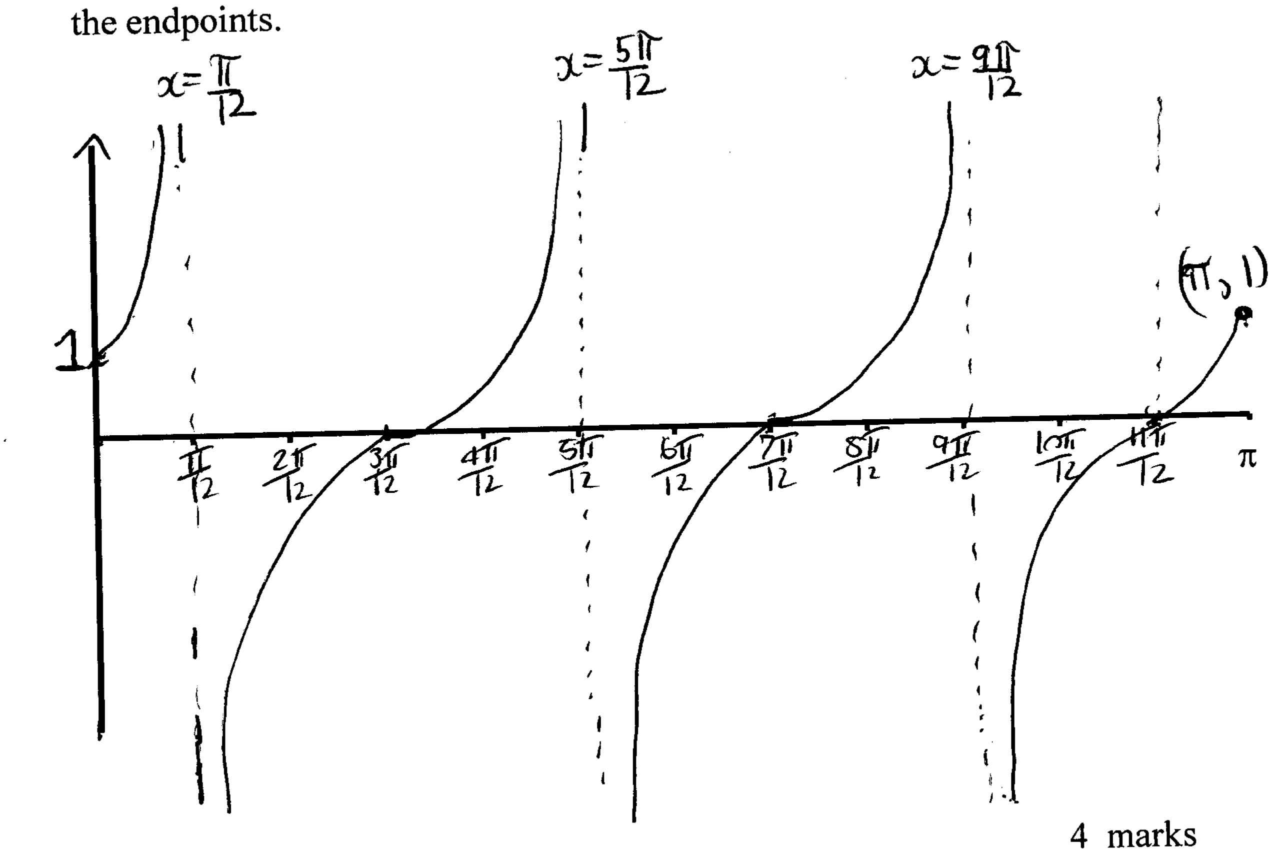
Period = $\frac{\pi}{3}$ Shift: $\frac{\pi}{4}$ to right

First asymptote: $x = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

Question 9

y-int: $h(0) = \tan\left(-\frac{3\pi}{4}\right) = 1$

a. Sketch the graph of the function: $h(x) = \tan(3\left(x - \frac{\pi}{4}\right))$ for the domain $0 \le x \le \pi$. State the equations of any asymptotes and the co-ordinates of the endpoints.



4 marks

b. i. Find the general formula which gives all the solutions to the equation:

$$\tan\left(3\left(x - \frac{\pi}{4}\right)\right) = \sqrt{3}.$$

Period = $\frac{\pi}{3}$

1st solution:

$3x - \frac{3\pi}{4} = \frac{\pi}{3}$

$3x = \frac{3\pi}{4} + \frac{\pi}{3}$

$\Rightarrow 3x = \frac{13\pi}{12}$

$\therefore x = \frac{13\pi}{36}$

$\Rightarrow$ General solution

$x = \frac{13\pi}{36} + \frac{\pi n}{3}$

ii. Use your formula from i above to determine the first **positive** solution of the above equation.

$n = -1$: $x = \frac{-\pi}{3} + \frac{13\pi}{36}$

$= \frac{\pi}{36}$

(1+1=2) marks

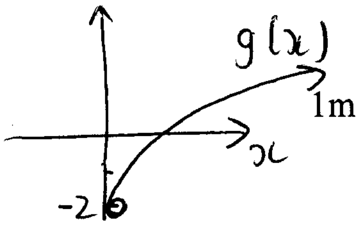
Total Question 9 = 6 marks

Section B: Analysis Question (11 marks)

Let $f(x) = \log_{10}(x + 2)$ and $g: (0,\infty) \to R, g(x) = \sqrt{x} - 2$

1. State the maximal domain of f.

$(-2, \infty)$



1mark

2. State the range of both f and g.

$\text{ran}(f) = \mathbb{R}$

$\text{ran}(g) = (-2,\infty)$

2 marks

3. Find f^{-1} and g^{-1} and state their domains.

dom(f)	ran(f)
(-2,∞)	ℝ
dom(f^{-1})	ran(f^{-1})
ℝ	(-2,∞)

$y = \log_{10}(x+2)$

$\downarrow$

$x = \log_{10}(y+2)$

$y + 2 = 10^x$

$f^{-1}(x) = 10^x - 2, \; x \in \mathbb{R}$

dom(g)	ran(g)
(0,∞)	(-2,∞)
dom(g^{-1})	ran(g^{-1})
(-2,∞)	(0,∞)

$y = \sqrt{x} - 2$

$x = \sqrt{y} - 2$

$y = (x+2)^2$

$g^{-1}(x) = (x+2)^2, \; x > -2$

3 marks

4. Find the composite function $(f \circ g)$ and explain why function is defined.

$f \circ g$ exists because $\text{ran}(g) \subseteq \text{dom}(f)$

since: $(-2,\infty) = \text{dom}(f)$ and $(-2,\infty) = \text{ran}(g)$.

$f \circ g(x) = f(g(x))$

$= \log_{10}(\sqrt{x} - 2 + 2)$

$= \log_{10}(\sqrt{x})$

2 marks

5. Evaluate $f(g(1000))$

$f \circ g(x) = \log_{10}(\sqrt{x}) = \frac{1}{2}\log_{10}x$

1 mark

$\therefore f \circ g(1000) = \frac{1}{2}\log_{10}(1000)$

$= \frac{3}{2}$

6. Let $h: D \rightarrow R, h(x) = \log_{10}(x+2)$ and $g: (0,\infty) \rightarrow R, g(x) = \sqrt{x} - 2$

i. Write down the rule of the function $g(h(x))$

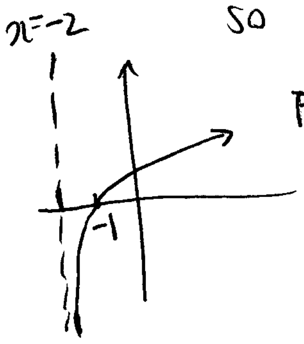
$g(h(x)) = \sqrt{\log_{10}(x+2)} - 2$

1 mark

ii. What is the biggest possible domain *D* for which the function $g(h(x))$ can be defined?

$g(h(x))$ is defined if $\text{ran}(h) \subseteq \text{dom}(g)$

$\text{dom}(g) = (0,\infty)$. Therefore $h(x)$ must be restricted so that its range is $(0,\infty)$



From the graph of $y = \log_{10}(x+2)$, we see that $\log_{10}(x+2) > 0$ if $x > -1$

$\therefore D = (-1,\infty)$

1 marks

Total Question 10= 11 marks