SOLUTIONS FOR MIDYEAR EXAM

Q1.

(a) $f(x) = x\log_e(x^2+5)$

Let $u = x$, $v = \log_e(x^2+5)$

$\frac{dy}{dx} = 1$, $\frac{dv}{dx} = \frac{1}{x^2+5} \times 2x = \frac{2x}{x^2+5}$

$\therefore f'(x) = x \times \frac{2x}{x^2+5} + 1 \times \log_e(x^2+5)$

$= \frac{2x^2}{x^2+5} + \log_e(x^2+5)$

(b) $y = \frac{\tan(2x)}{e^{2x}}$

$\therefore y = e^{-2x}\tan(2x)$

$u = \tan(2x)$, $u' = 2\sec^2 2x$ $\quad v = e^{-2x}$, $v' = -2e^{-2x}$

$\therefore \frac{dy}{dx} = -2e^{-2x}\tan(2x) + e^{-2x} \times 2\sec^2 2x$

$= 2e^{-2x}(\sec^2(2x) - \tan(2x))$

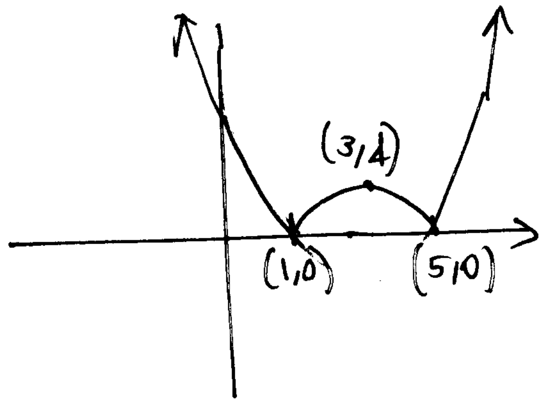
$\left.\frac{dy}{dx}\right|_{x=0} = 2e^0\left(\frac{1}{\cos^2 0} - \tan(0)\right)$

$= 2 \times 1 \times (1 - 0)$

$= 2$

Q2. (a) $f: \mathbb{R} \to \mathbb{R}, f(x) = |x^2 - 6x + 5|$

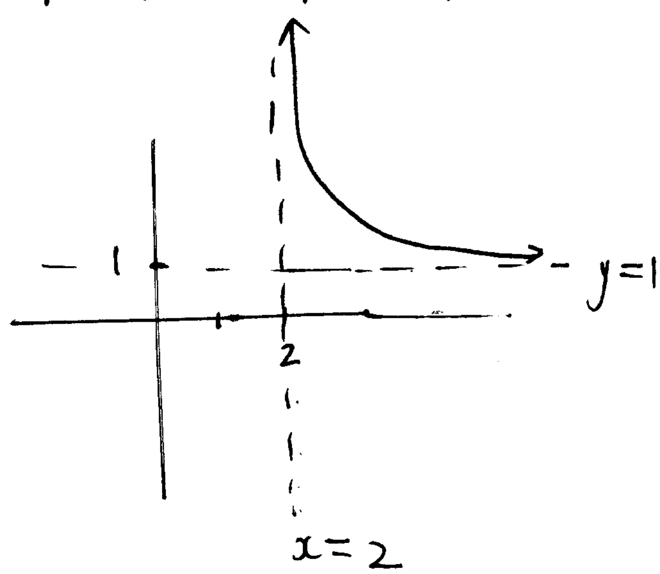
$= |(x-5)(x-1)|$



(b) Domain of derivative $= \mathbb{R} \setminus \{1, 5\}$

(c) $f'(x) > 0$ for $1 < x < 3 \cup x > 5$

Q3. (a)



(b)

dom(h)	ran(h)
$(2, \infty)$	$(1, \infty)$
dom(h^{-1})	ran(h^{-1})
$(1, \infty)$	$(2, \infty)$

$x = \frac{1}{y-2} + 1$

$x - 1 = \frac{1}{y-2}$

$\therefore y - 2 = \frac{1}{x-1}$

$\therefore h^{-1}(x) = \frac{1}{x-1} + 2, x \in (1, \infty).$

Q4. (a)

$$2y + 2x + \text{Semi-circumference} = 100$$

$$\text{Semicircumence} = \frac{2\pi r}{2} = \pi r$$
$$= \pi x$$

$$\therefore \quad 2y + 2x + \pi x = 100$$
$$\therefore \quad 2y = 100 - \pi x - 2x$$
$$y = 50 - \frac{\pi x}{2} - x$$

(b) Surface area:

$$A = 2xy + \frac{\pi x^2}{2}$$
$$= 2x\left(50 - \frac{\pi x}{2} - x\right) + \frac{\pi x^2}{2}$$
$$= 100x - \pi x^2 - 2x^2 + \frac{\pi x^2}{2}$$
$$= 100x - \frac{\pi x^2}{2} - 2x^2$$
$$= 100x - \frac{x^2}{2}(\pi + 4)$$

(c) For a maximum, $A'(x) = 0$

$$A'(x) = 100 - \pi x - 4x$$
$$= 100 - x(\pi + 4)$$

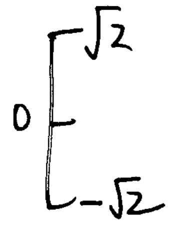
Let $A'(x) = 0 \quad \therefore \quad 0 = 100 - (\pi + 4)x$

$$\therefore \quad x = \frac{100}{\pi + 4}$$

Q5.

$h(x) = \sqrt{2}\sin(2x)$

(a) $\therefore$



Range: $h \in [-\sqrt{2}; \sqrt{2}]$

(b) $\sqrt{2}\sin(2x) = -1$

$$\sin(2x) = \frac{-1}{\sqrt{2}}$$

Period of $\sin(2x)$ is π.

$$\sin(2x) = \frac{-1}{\sqrt{2}} \quad \text{for } 0 \leq x \leq 2\pi$$

gives: $2x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

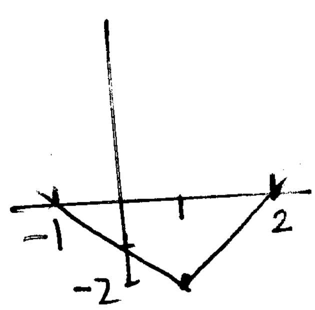
$\therefore x = \frac{5\pi}{8}$ or $\frac{7\pi}{8}$

$\therefore$ General solution:

$$x = n\pi + \frac{5\pi}{8} \quad \text{or} \quad x = n\pi + \frac{7\pi}{8}$$

SECTION B: MULTIPLE CHOICE

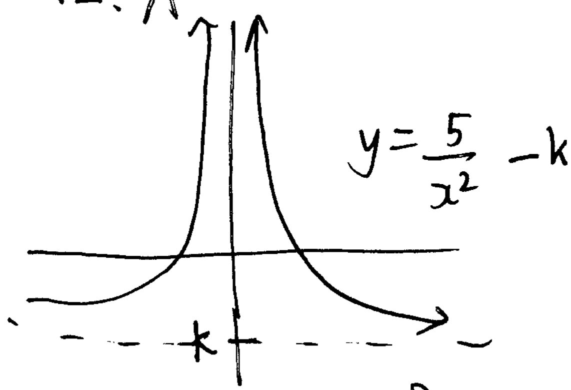
Q1. A



$f(x) = |x-1| - 2, \; x \in [-1, 2]$

Range: $[-2, 0)$

Q2. A



Domain: $\mathbb{R} \setminus \{0\}$

Range: $(-k, \infty)$

Q3. D

$4\sin(2x) + 1 = 3.6$

Put CAS in degree mode and solve for $0 \leq x \leq 360$.

gives $x = 20.27, 69.73, 200.27, 249.73$

Q4. C

Period $= 4\pi$ $\quad \therefore 4\pi = \frac{2\pi}{n} \quad \therefore n = \frac{1}{2}$

Median axis: $y = 3$

Cosine shape ; Amp $= 3$

$\therefore y = 3\cos\left(\frac{x}{2}\right) + 3$

Q5. D

$$\left(\frac{x^3 y^{-2}}{x^2 y^5}\right)^{-2} \times \sqrt[3]{\frac{y^6}{x^{12}}} = \frac{x^{-6} y^4}{x^4 y^{-10}} \times \frac{y^2}{x^4}$$

$$= \frac{x^{-6} y^6}{x^8 y^{-10}} = \frac{y^{16}}{x^{14}}$$

Q6. E

Averge rate of change

$$= \frac{g\left(\frac{\pi}{4}\right) - g(0)}{\frac{\pi}{4} - 0}$$

$$= \frac{\left|\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\right| - \left|\cos\left(-\frac{\pi}{3}\right)\right|}{\frac{\pi}{4} - 0}$$

$$= \frac{\left|\cos\left(\frac{\pi}{6}\right)\right| - \cos\left(\frac{-\pi}{3}\right)}{\frac{\pi}{4}}$$

$$= \frac{\frac{\sqrt{3}}{2} - \frac{1}{2}}{\frac{\pi}{4}} = \frac{2(\sqrt{3}-1)}{\pi}$$

Q7. B

$$\lim_{x \to -1} \frac{x^2 - 4x - 5}{x+1}$$

$$= \lim_{x \to -1} \frac{(x+1)(x-5)}{(x+1)}$$

$$= \lim_{x \to -1} (x-5) = -6$$

Q8. D

$$y = \frac{a}{2x-5} + 1 = a(2x-5)^{-1} + 1$$

$$\therefore \frac{dy}{dx} = a \times -1 \times (2x-5)^{-2} \times 2$$

$$= \frac{-2a}{(2x-5)^2}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=3} = \frac{-2a}{1^2} = -2a.$$

Q8. [Cont].

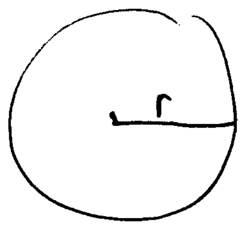
Since $m_{NORMAL} = \frac{1}{2}$, $M_{TANGENT} = -2$

$\therefore -2a = -2$

$\therefore a = 1.$

Q9. E

Q10. C



Variables: V, r, t

Given: $\frac{dv}{dt} = -11.52\pi$

$\frac{dr}{dt} = -2$ (snapshot). Find: r

$$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt}$$

$V = \frac{4\pi \pi r^3}{3}$

$\therefore \frac{dV}{dr} = 4\pi r^2$

$-11.52\pi = 4\pi r^2 \times -2$

$\therefore 11.52\pi = 4\pi r^2 \times 2$

$\therefore r^2 = \frac{11.52}{8}$ $\therefore r = \sqrt{\frac{11.52}{8}}$

$r = 1.2$

Q11. D

cosine function

Period $= \frac{2\pi}{3}$ $\therefore \frac{2\pi}{n} = \frac{2\pi}{3}$ $\therefore n = 3$

$-2 \left[\begin{array}{l} 1 \\ b \\ -5 \end{array}\right.$ $b = \frac{1+-5}{2} = -2$ $\therefore$ Amplitude $= 3$; Vertical shift: -2

$\therefore y = 3\cos(3x) - 2$

This answer isn't there — it must be either D or E.

On CAS: $\sin\left(3x+\frac{\pi}{2}\right)$ simplifies to $\cos(3x)$

and $\cos(3x-\pi)$ simplifies to $-\cos(3x)$.

$\therefore \; y = 3\cos(3x) - 2 = 3\sin\left(3x+\frac{\pi}{2}\right) - 2$

$\therefore$ D is correct.

Q12. E

A is not correct since $\tan\left(\frac{\pi}{2}+\theta\right) = \dfrac{\sin\left(\frac{\pi}{2}+\theta\right)}{\cos\left(\frac{\pi}{2}+\theta\right)}$

$= \dfrac{\cos\theta}{-\sin\theta}$

and $\tan\left(\frac{\pi}{2}-\theta\right) = \dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}$

$= \dfrac{\cos\theta}{\sin\theta}$

B is not correct for same reason as A.

C is not correct: $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

$\therefore \cos\left(\frac{\pi}{2}+\theta\right) \neq \cos\left(\frac{\pi}{2}-\theta\right)$

D is not correct for same reason.

$\therefore$ E must be correct!

Check: $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$ and $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

$\therefore \sin\left(\frac{\pi}{2}+\theta\right) = \sin\left(\frac{\pi}{2}-\theta\right)$

$-\sin\left(\theta-\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

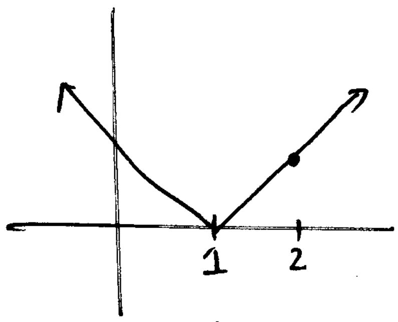
Q13. C

By Chain Rule, $\frac{d}{dx}(h(\cos 2x))$

$= h'(\cos 2x) \times -2\sin 2x$

Q14. D

$y = |1-x| = |x-1|$



At $x=2$, rate of change of y with respect to x

= gradient of line

$= +1.$

Q15. B

$$\begin{bmatrix} 5 & a-3 \\ a & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix}$$

For no unique solution, $\det(A) = 0$

$\therefore 5 \times 2 - a(a-3) = 0$

$10 - a^2 + 3a = 0$

$a^2 - 3a - 10 = 0$

$(a-5)(a+2) = 0$

$\therefore a = 5, -2$

If $a=5$: $5x + 2y = 1$

$5x + 2y = 5$

These are parallel lines

$\therefore$ No solution

If $a=-2$: $5x - 5y = 1$

$-2x + 2y = -2$

$\therefore$ These are the same line.

$\therefore$ Infinitely many solutions.

Sec C.

(a) $f(0) = 0$

$\therefore e^0 - b = 0$

$\therefore b = 1$

(b) $f(x) = e^{ax} - 1$

$\therefore f'(x) = ae^{ax}$

$f'(1) = 2e^2$

$\therefore ae^a = 2e^2$

$\therefore a = 2$

(c)

- Reflect in x-axis
- Translate in negative y-direction by 1 unit

(ii) $f(x) = e^{2x} - 1$

$\therefore -f(x) = -e^{2x} + 1$

$-f(x) - 1 = -e^{2x}$

$\therefore g(x) = -e^{2x}$

$e^{2x} - 1 \Rightarrow -(e^{2x} - 1) = -e^{2x} + 1 \Rightarrow -e^{2x} + 1 - 1$

$= -e^{2x}$

(d) $y = e^{2x} - 1$ ①

$y = -e^{2x}$ ②

$e^{2x} - 1 = -e^{2x}$

$2e^{2x} = 1$

$e^{2x} = \frac{1}{2}$

$2x = \log_e\left(\frac{1}{2}\right)$

$x = \frac{1}{2}\log_e\left(\frac{1}{2}\right) = -\frac{1}{2}\log_e 2$

$\therefore c = -\frac{1}{2}\log_e 2$

(e) $f(x) = -f\left(\frac{x}{2}\right)$

$\therefore e^{2x} - 1 = -e^{2 \times \frac{x}{2}} + 1$

$\therefore e^{2x} + e^{x} - 2 = 0$

Let $e^{x} = p$

$p^2 + p - 2 = 0$

$(p+2)(p-1) = 0$

$p = -2, 1$

$\therefore e^{x} = -2$ or $e^{x} = 1$

$\downarrow$ $\downarrow$

$\therefore$ No solution $x = 0$

$\therefore x = 0$

Q2. $f(x) = \cos(2x)$

(a) $f'(x) = -2\sin(2x)$

(b) $-2\sin(2x) = -1$

$\sin(2x) = \frac{1}{2}$

$\therefore 2x = \frac{\pi}{6}, \frac{5\pi}{6}$ $\qquad$ $0 \leq x \leq \frac{\pi}{2}$

$0 \leq 2x \leq \pi$

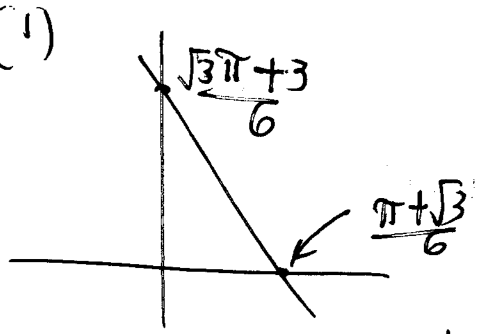
$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}$

When $x = \frac{\pi}{12}$, $f(\frac{\pi}{12}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

When $x = \frac{5\pi}{12}$, $f(\frac{5\pi}{12}) = \cos(\frac{5\pi}{6})$ $\quad -\frac{\sqrt{3}}{2}$

$\therefore (\frac{\pi}{12}, \frac{\sqrt{3}}{2})$ and $(\frac{5\pi}{12}, -\frac{\sqrt{3}}{2})$

(c)(i)



$$\text{Gradient} = \frac{\frac{\pi+\sqrt{3}}{6} - 0}{0 - \left(\frac{\sqrt{3}\pi+3}{6}\right)}$$

$$\frac{\frac{\sqrt{3}\pi+3}{6}}{-\frac{(\pi+\sqrt{3})}{6}}$$

$$= \frac{\sqrt{3}(\pi+\sqrt{3})}{-(\pi+\sqrt{3})} = -\sqrt{3}$$

Q2. (c) (ii)

$f'(x) = -\sqrt{3}$

$\therefore -2\sin 2x = -\sqrt{3}$

$\sin(2x) = \frac{\sqrt{3}}{2}$, $0 \leq 2x \leq \pi$

$\therefore 2x = \frac{\pi}{3}, \frac{2\pi}{3}$

$\therefore x = \frac{\pi}{6}, \frac{\pi}{3}$

When $x = \frac{\pi}{6}$, $f(\frac{\pi}{6}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$

$\therefore (\frac{\pi}{6}, \frac{1}{2})$ is

$x = \frac{\pi}{3}$, $f(\frac{\pi}{3}) = \cos(\frac{2\pi}{3}) = -\frac{1}{2}$

$\therefore (\frac{\pi}{3}, -\frac{1}{2})$

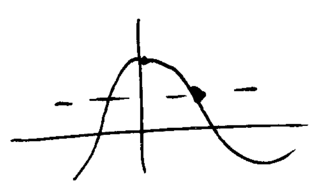
(d)

$|\cos(2x)|$ has a maximum value of 1 and a minimum value of 0.

(e) $g(x) = \cos(2x)$

$\cos(2x) = \frac{1}{2}$ Period $= \frac{2\pi}{2} = \pi$

$\cos(2x) = \frac{1}{2}$



$\therefore 2x = \frac{\pi}{3}, -\frac{\pi}{3}$

$x = \frac{\pi}{6}, -\frac{\pi}{6}$

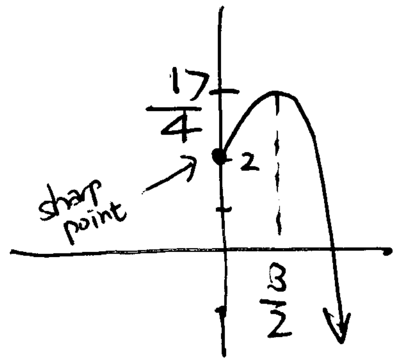
General solution: $n\pi \pm \frac{\pi}{6}$

Q3. $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $f(x) = -x^2 + 3x + 2$

(a) $f'(x) = -2x + 3$ where $x > 0$

$f'(x) > 0$ if $-2x + 3 > 0$

$0 < x < \frac{3}{2}$



(b) $f\left(\frac{3}{2}\right) = -\frac{9}{4} + \frac{9}{2} + 2 = \frac{-9}{4} + \frac{18}{4} + \frac{8}{4} = \frac{17}{4}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$$

$x' = x + c$

$y' = -y + d$

$(x', y') = (0, 0)$

$(x, y) = \left(\frac{3}{2}, \frac{17}{4}\right)$

$\therefore 0 = \frac{3}{2} + c \quad \therefore c = \frac{-3}{2}$

$0 = -\frac{17}{4} + d \quad \therefore d = \frac{17}{4}$

(c) It must be dilated away from y-axis (since a dilation away from x-axis will not change x-intercepts)

x-intercept of $f(x)$: $0 = -x^2 + 3x + 2$

$\therefore x^2 - 3x - 2 = 0$

$x = \frac{3 \pm \sqrt{9+8}}{2}$

$= \frac{3 + \sqrt{17}}{2}$ (since $x > 0$)

For dilation from y-axis

$$x' = kx$$

$$\therefore 2(3+\sqrt{17}) = \frac{k(3+\sqrt{17})}{2}$$

$$\therefore k = 4.$$

(d) (i) $h(x) = |x|$

$g(x) = f(h(x))$

(i) $g(x) = -|x|^2 + 3|x| + 2$

$= -x^2 + 3|x| + 2$

(ii) $\text{ran}(h) = [0, \infty)$

$\text{dom}(f) = [0, \infty)$

$\therefore \text{ran}(h) \subseteq \text{dom}(f)$

(iv) $g(x) = -x^2 + 3x + 2, x > 0$

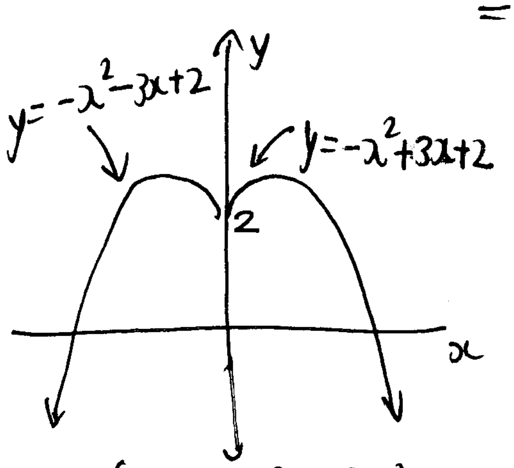
$= -x^2 - 3x + 2, x < 0$

$= 2, x = 0.$

$|x| = x, x > 0$

$= -x, x < 0$

$= 0, x = 0$



$\therefore g'(x) = -2x + 3, x > 0$

$= -2x - 3, x < 0$

undefined at $x = 0$.

Graph of $g(x)$

shows a sharp point at $x = 0$

$\therefore$ Derivative is undefined there.

(iv)