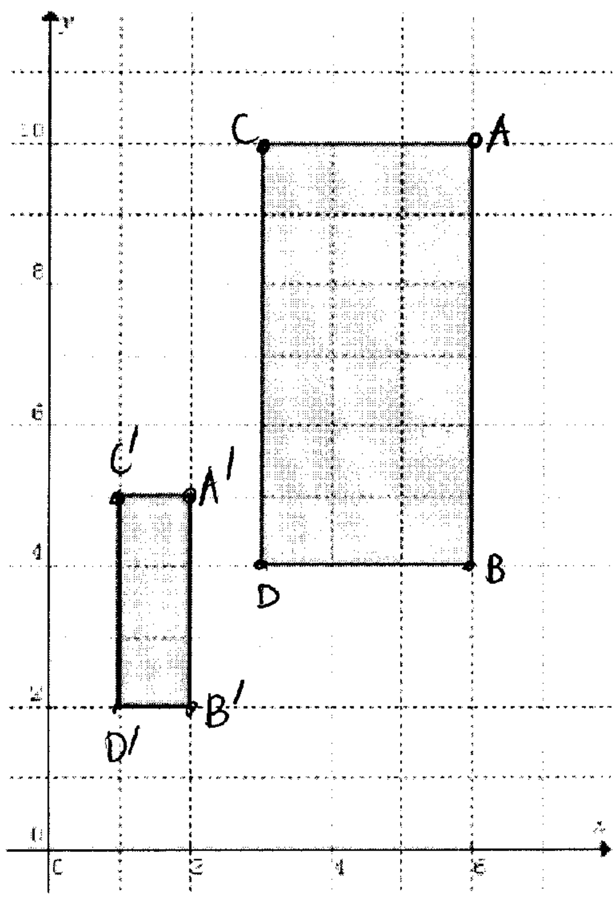
SOLUTIONS.

Part A: Multiple Choice questions — 10 x 1 mark each = 10 marks

Question 1



Transformation T changes the large rectangle to the small rectangle. T is defined by

A. $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{1}{3}&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}2\\2\end{bmatrix}$

B. $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}2\\2\end{bmatrix}$

C. $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{1}{3}&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}-\begin{bmatrix}4\\5\end{bmatrix}$

D. $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{1}{3}&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

E. $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$(6,10) \to (2,5)$

$(6,4) \to (2,2)$

$(3,10) \to (1,5)$

$(3,4) \to (1,2)$

$\therefore (x,y) \to (\frac{1}{3}x, \frac{1}{2}y)$

$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}\frac{1}{3}&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Question 2

If $f:\left(-\infty,\frac{1}{2}\right]\to R$, where $f(x)=\sqrt{(1-2x)}$ and $g:R\setminus\left\{\frac{1}{2}\right\}\to R$, where $g(x)=\frac{1}{1-2x}$ then the domain of fg is

A. R^-

B. $R\setminus\left\{\frac{1}{2}\right\}$

C. $\left(-\infty,\frac{1}{2}\right)$

D. $\left(-\infty,\frac{1}{2}\right]$

E. R

~~dom f(x)~~ =

$fg(x) = \frac{\sqrt{1-2x}}{1-2x} = \frac{1}{\sqrt{1-2x}}$

$\therefore$ domain of $fg = \left(-\infty, \frac{1}{2}\right)$

Question 3

Function f is defined by $f(x)=(x-a)^2-b$, $x\in R$.

f^{-1}, the inverse function of f.

A. does not exist

B. is defined by $f^{-1}(x)=a\pm\sqrt{x+b}$, $x\in R$

C. is defined by $f^{-1}(x)=a\pm\sqrt{x+b}$, $x\geq -b$

D. is defined by $f^{-1}(x)=a\pm\sqrt{x-b}$, $x\in R$

E. is defined by $f^{-1}(x)=-a\pm\sqrt{x-b}$, $x\geq b$

$f(x)=(x-a)^2-b, x\in R$ is a parabola and is not one-to-one. Therefore, f^{-1} does not exist.

Question 4

f is the function defined by $\frac{1}{x^2+2}, x\in R$.

A suitable restriction for f to becomes f^*, such that f^{*-1} exists, would be:

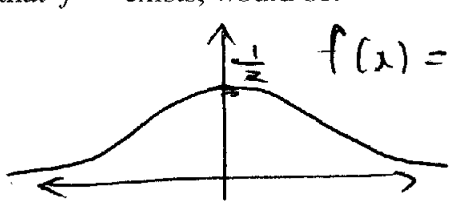
A. $f^*:[-1,1]\to R, f^*(x)=\frac{1}{x^2+2}$

B. $f^*:R\to R, f^*(x)=\frac{1}{x^2+2}$

C. $f^*:[-2,2]\to R, f^*(x)=\frac{1}{x^2+2}$

D. $f^*:[0,\infty)\to R, f^*(x)=\frac{1}{x^2+2}$

E. $f^*:[-1,\infty)\to R, f^*(x)=\frac{1}{x^2+2}$



$f(x)=\frac{1}{x^2+2}$

We must restrict the domain of f to make f^* one-to-one. D provides such a restriction.

Question 5

The implied (largest possible) domain for the function with the rule $y = \frac{1}{\sqrt{2-x}}$ is:

A. $R\setminus\{2\}$
B. $(-\infty, 2)$
C. $(2, \infty)$
D. $(-\infty, 2]$
E. R^+

$2 - x > 0$
$\therefore x < 2$

Question 6

A sequence of transformations which maps $y = x^2$ to $y = 2x^2 - 4x + 5$ is:

$y = 2(x^2 - 2x) + 5$
$y = 2(x^2 - 2x + 1) + 5 - 2$
$y = 2(x-1)^2 + 3$

A a translation where $(x, y) \to (x + 1, y + 3)$ followed by a dilation of $\frac{1}{2}$ from the x-axis

B a translation where $(x, y) \to (x + 1, y + 3)$ followed by a dilation of 2 from the x-axis

C a dilation of $\frac{1}{2}$ from the x-axis then a translation where $(x, y) \to (x + 1, y + 3)$

D a dilation of 2 from the x-axis then a translation where $(x, y) \to (x + 1, y + 3)$

E a dilation of $\frac{1}{2}$ from the x-axis then a translation where $(x, y) \to (x - 1, y - 3)$

- dilation factor 2 away from x
- translate 1 unit to right
- translate 3 units up.

Question 7

A curve has equation $y = f(x)$. The following transformations are applied to the curve in the given order:

- a reflection in the y-axis
- a dilation of factor 2 from the x-axis
- a translation of 2 units in the positive direction of the y-axis

$f(x)$
$\downarrow$
$f(-x)$
$\downarrow$
$2f(-x)$
$\downarrow$
$2f(-x) + 2$

The equation of the resulting curve is:

A $y = -2f(x) - 2$
B $y + 2 = -2f(x)$
C $y = 2f(-x) + 2$
D $y = -2f(-x) + 2$
E $y = f(x - 2) + 2$

Question 8

The domain of the function $y=\sqrt{\frac{x-3}{x-6}}$ is:

$\left(\frac{x-3}{x-6}\right) \geq 0 \therefore \frac{x-3}{x-6} \times (x-6)^2 > 0$

$\therefore (x-3)(x-6) > 0, x \neq 6$

$x: x < 3 \cup x > 6$

A. $[3,6)$
B. $(6,\infty)$
C. $[3,\infty)$
D. $R\backslash\{6\}$
E. $(-\infty,3]\cup(6,\infty)$

Question 9

The function $f(x)=2x^3-12x+5$ has a turning point at $\left(\sqrt{2},5-8\sqrt{2}\right)$. It follows that the function $g(x)=3+f(2-x)$ has a turning point at

A. $\left(2+\sqrt{2},8-8\sqrt{2}\right)$
B. $\left(\sqrt{2}-2,8\sqrt{2}-2\right)$
C. $\left(\sqrt{2}+2,8\sqrt{2}-2\right)$
D. $\left(2-\sqrt{2},8-8\sqrt{2}\right)$
E. $\left(3+\sqrt{2},8\sqrt{2}-7\right)$

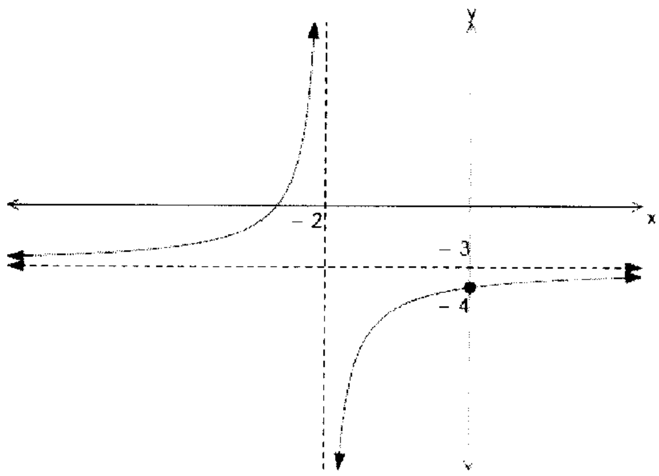
$g(x) = f(-(x-2)) + 3$

$(x, y) \rightarrow (-x, y) \rightarrow (-x+2, y+3)$

$\therefore (\sqrt{2}, 5-8\sqrt{2}) \rightarrow (-\sqrt{2}+2, 8-8\sqrt{2})$

Question 10

The graph of the function with equation $y = \dfrac{1}{Ax+B} - C$, where A, B and C are real constants, is shown below:



The values of A, B and C respectively are:

A. $A = -\frac{1}{2}$ $\quad B = -1$ $\quad C = 3$

B. $A = -1$ $\quad B = -2$ $\quad C = -3$

C. $A = 1$ $\quad B = 1$ $\quad C = 3$

D. $A = -2$ $\quad B = 1$ $\quad C = -3$

E. $A = 1$ $\quad B = -2$ $\quad C = 3$

$y = \dfrac{1}{Ax+B} - 3 \quad (\therefore C = 3)$

When $x = 0$, $y = -4$

$-4 = \dfrac{1}{B} - 3$

$\therefore -1 = \dfrac{1}{B}$

$\therefore B = -1$

By elimination, only option A can be correct!

However, to confirm this: we know that there is a vertical asymptote at $x = -2$

$\therefore -2A + B = 0$

But $B = -1$

$\therefore -2A - 1 = 0$

$\therefore A = -\frac{1}{2}$

Part B: Short answer questions: (5 questions = 7+4+10+6+3=30marks)

Question 1

Let $f : R \setminus \{-2\} \to R,\ f(x) = \dfrac{3}{x+2} - 1$.

a. Find f^{-1}, the inverse function of f. State its domain.

OUTCOME1	OUTCOME3	TOTAL MARKS
3	0	3

dom(f) $\mathbb{R}\setminus\{-2\}$ ran(f) $\mathbb{R}\setminus\{-1\}$

dom(f^{-1}) $\mathbb{R}\setminus\{-1\}$ ran(f^{-1}) $\mathbb{R}\setminus\{-2\}$

$$y = \frac{3}{x+2} - 1$$
$$\downarrow$$
$$x = \frac{3}{y+2} - 1$$
$$x+1 = \frac{3}{y+2}$$
$$y+2 = \frac{3}{x+1}$$

$$\therefore f^{-1}(x) = \frac{3}{x+1} - 2,\ x \in \mathbb{R}\setminus\{-1\}$$

b. Let $g : [2, \infty) \to R,\ g(x) = x^2 - 4$.

Find the rule for $f(g(x))$. State its domain

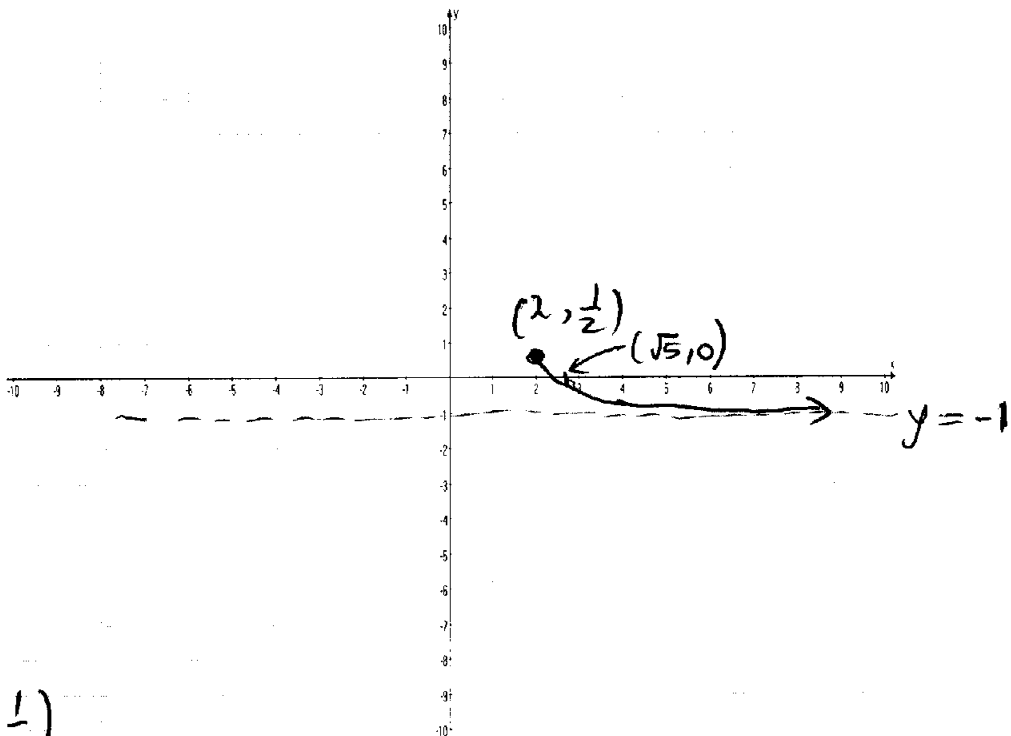
OUTCOME1	OUTCOME3	TOTAL MARKS
2	0	2

$$f(g(x)) = \frac{3}{g(x)+2} - 1$$
$$= \frac{3}{x^2-4+2} - 1$$
$$= \frac{3}{x^2-2} - 1$$

Domain of $f(g(x))$ = dom(g) = $[2, \infty)$.

c. Sketch $f(g(x))$ for its domain (use CAS to check your sketch)

OUTCOME1	OUTCOME3	TOTAL MARKS
2	0	2



Endpoint: $\left(2, \frac{1}{2}\right)$

$$\frac{3}{x^2-2} - 1 = 0$$

$$\frac{3}{x^2-2} = 1$$

$$3 = x^2 - 2$$

$$x^2 = 5$$

$$x = \sqrt{5}$$

$\therefore$ x-int: $(\sqrt{5}, 0)$

Total question 1 = (3+2+2=7 marks)

Question 2

For the simultaneous linear equations

$$3x + ky = 2$$
$$(k-2)x + y = k-1$$

k is a real constant.

a. Find the value of k for which there are no solutions.

OUTCOME1	OUTCOME3	TOTAL MARKS
3	0	3

$$\begin{bmatrix} 3 & k \\ k-2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ k-1 \end{bmatrix}$$

For no unique solution, determinant $= 0$

$\therefore 3 - k(k-2) = 0$

$3 - k^2 + 2k = 0$

$\therefore k^2 - 2k - 3 = 0$

$(k-3)(k+1) = 0$

$k = 3, -1.$

If $k = 3$: eqns are: $3x + 3y = 2$

$x + y = 2$ which gives no solution

If $k = -1$,

$3x - y = 2$

$-3x + y = -2$

gives infinitely many solutions

$\therefore \boxed{k = -1}$

b. For what values of k does a unique solution exist?

OUTCOME1	OUTCOME3	TOTAL MARKS
1	0	1

For a unique solution, determinant $\neq 0$

$\therefore k \in \mathbb{R} \setminus \{3, -1\}$

Total question 2 = (3+1=4 marks)

Question 3

Let $f:(-2,\infty)\to R, f(x)=\dfrac{4}{(x+2)^2}-4$

a. Find the inverse function $f^{-1}(x)$, stating its domain and range.

OUTCOME1	OUTCOME3	TOTAL MARKS
3	0	3

dom(f)	ran(f)
$(-2,\infty)$	$(-4,\infty)$
dom(f^{-1})	ran(f^{-1})
$(-4,\infty)$	$(-2,\infty)$

$$y=\frac{4}{(x+2)^2}-4$$

$$x=\frac{4}{(y+2)^2}-4$$

$$\frac{x+4}{4}=\frac{1}{(y+2)^2}$$

$$(y+2)^2=\frac{4}{x+4}$$

$$y+2=\pm\sqrt{\frac{4}{x+4}}$$

Since the range of f^{-1} is $(-2,\infty)$ we select the positive square root

$$\therefore f^{-1}(x)=-2+\sqrt{\frac{4}{x+4}},$$

$$x\in(-4,\infty)$$

b. On the same diagram, sketch the graphs of $f(x)$ and $f^{-1}(x)$. Show all axes intercepts, point of intersection between the two graphs and the equations of any asymptotes. (Use CAS to check)

OUTCOME1	OUTCOME3	TOTAL MARKS
3	1	4

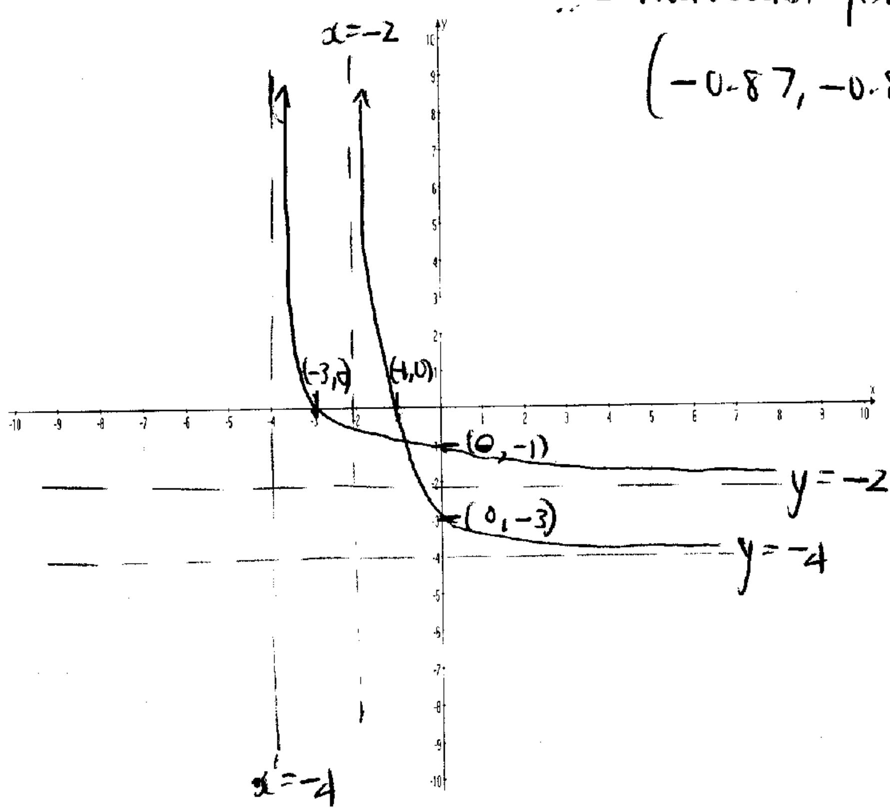
Point of Intersection: $y = x$ ①

$y = -2 + \sqrt{\dfrac{4}{x+4}}$ ②

Solving: $x = -0.8696$

$\therefore$ Intersection point

$(-0.87, -0.87)$



c- Show that $f\left(f^{-1}(x)\right) = x$. Hence sketch $f\left(f^{-1}(x)\right)$ for its domain

OUTCOME1	OUTCOME3	TOTAL MARKS
3	0	3

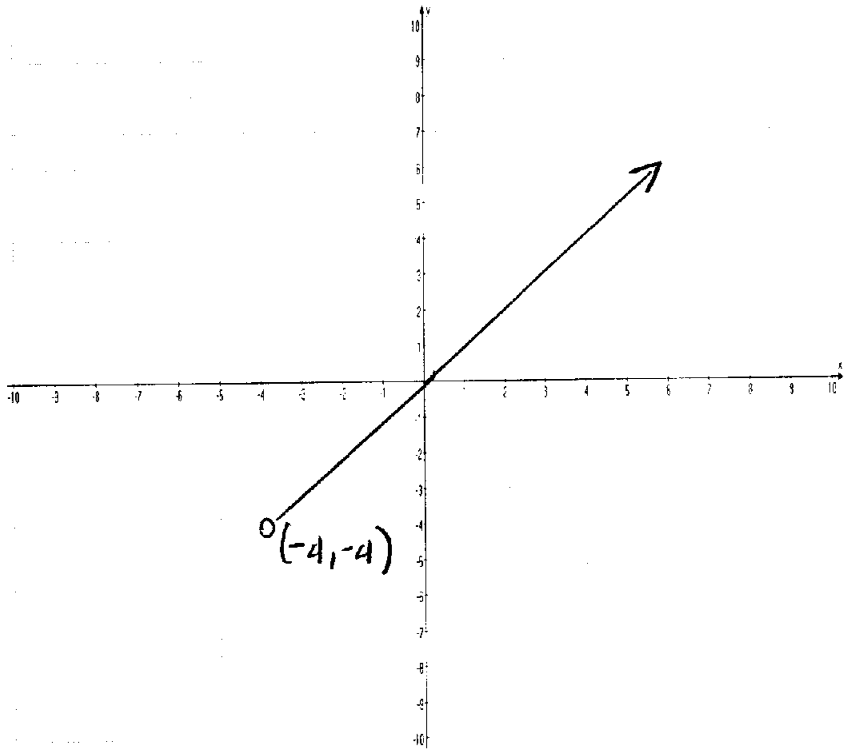
$$f\left(f^{-1}(x)\right)$$

$$= \frac{4}{\left(-2+\sqrt{\frac{4}{x+4}}+2\right)^2} - 4$$

$$= \frac{4}{\frac{4}{x+4}} - 4$$

$$= x+4-4$$

$$= x, \text{ where } x \in (-4, \infty)$$



Total question 3 = (3+4+3=10 marks)

Question 4

Two functions are $f(x) = 1 + \sqrt{x+1}$ and $g(x) = 2\sqrt{x}$.

a- Find the coordinates of the intersection of the two functions (*show working out. CAS answer only is not gaining full mark*)

OUTCOME1	OUTCOME3	TOTAL MARKS
3	0	3

$$1+\sqrt{x+1} = 2\sqrt{x}$$

$$\sqrt{x+1} = 2\sqrt{x} - 1$$

$$\left(\sqrt{x+1}\right)^2 = \left(2\sqrt{x}-1\right)^2$$

$$x+1 = 4x - 4\sqrt{x} + 1$$

$$-3x = -4\sqrt{x}$$

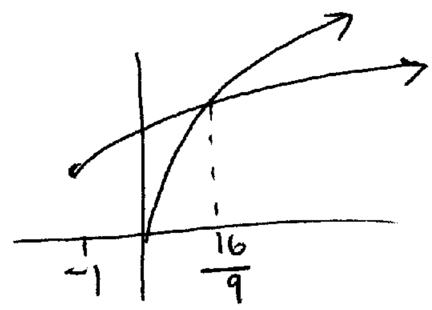
$$3x = 4\sqrt{x}$$

$$\therefore 9x^2 = 16x$$

$$9x^2 - 16x = 0$$

$$x(9x-16) = 0$$

$$\therefore x = 0, \frac{16}{9}$$



$x = 0$ must be rejected

If $x = \frac{16}{9}$, $y = 2\sqrt{\frac{16}{9}} = \frac{8}{3}$

or : If $x = \frac{16}{9}$, $y = 1 + \sqrt{\frac{16}{9}+1} = 1 + \sqrt{\frac{25}{9}} = \frac{8}{3}$

$\therefore$ Intersection point: $\left(\frac{16}{9}, \frac{8}{3}\right)$

b- The two functions undergo the following transformations in order as follows:

Reflection in y-axis,

4 units down;

2 units left;

Vertical dilation by factor of $\frac{1}{2}$;

Horizontal dilation by factor $\frac{1}{2}$

Find the coordinates of the intersection of the transformed functions. *(Hint: map the coordinates you have found in a according the given transformation in b.)*

OUTCOME1	OUTCOME3	TOTAL MARKS
3	0	3

$$(x,y) \rightarrow (-x, y) \rightarrow (-x, y-4) \rightarrow (-x-2, y-4)$$

$$\downarrow$$

$$\left(\tfrac{1}{2}(-x-2), \tfrac{1}{2}(y-4)\right) \leftarrow \left(-x-2, \tfrac{1}{2}(y-4)\right)$$

$$\therefore \left(\frac{16}{9}, \frac{8}{3}\right) \rightarrow \left(\frac{1}{2}\left(-\frac{16}{9}-2\right), \frac{1}{2}\left(\frac{8}{3}-4\right)\right)$$

$$= \left(\frac{1}{2} \times \frac{-34}{9}, \frac{1}{2} \times \frac{-4}{3}\right)$$

$$= \cancel{\left(\frac{-17}{18}\right.}, \left(-\frac{17}{9}, \frac{-2}{3}\right).$$

Co-ordinates are: $\left(\frac{-17}{9}, \frac{-2}{3}\right)$

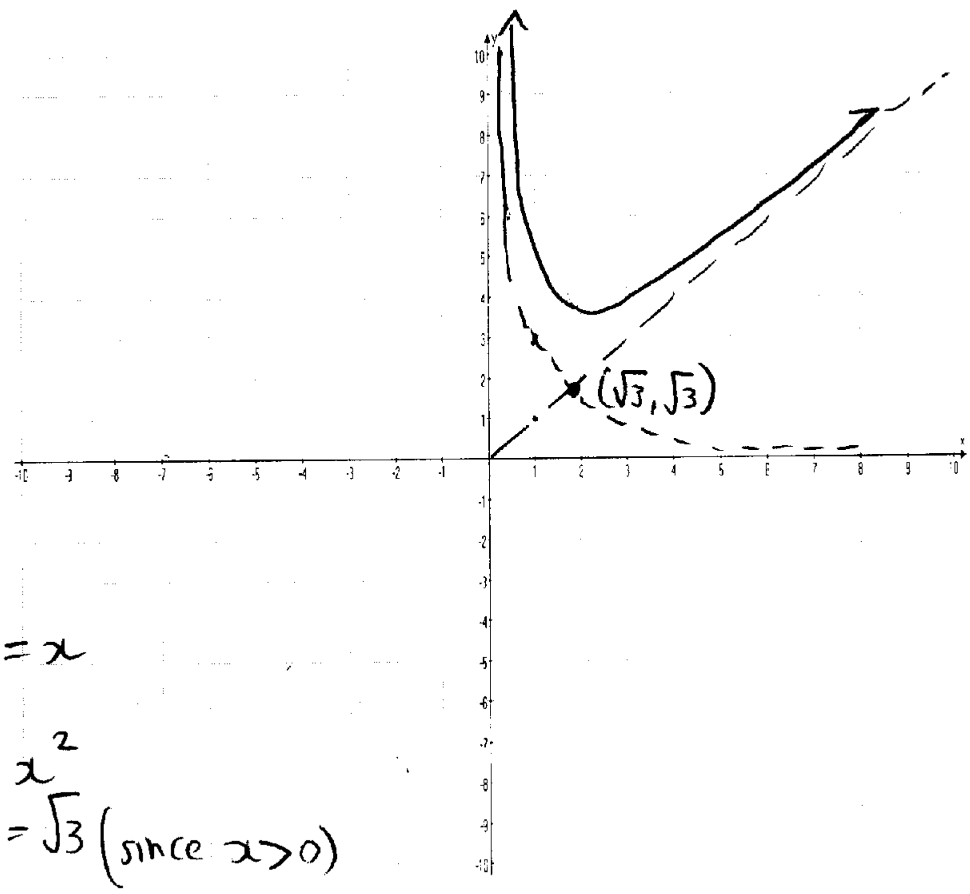
Total question 4=(3+3= 6 marks)

Question 5

a. if $f: R^{+} \to R, f(x) = \dfrac{3}{x}$ and $g: R^{+} \cup \{0\} \to R, g(x) = x$

Sketch the graph of $f(x) + g(x)$ on the axes below. Show any asymptotes and any points of intersection between $f(x)$ and $g(x)$. (*CAS can only be used as to check on your finding*)

OUTCOME1	OUTCOME3	TOTAL MARKS
3	0	3



$\dfrac{3}{x} = x$

$3 = x^2$

$\therefore x = \sqrt{3}$ (since $x > 0$)

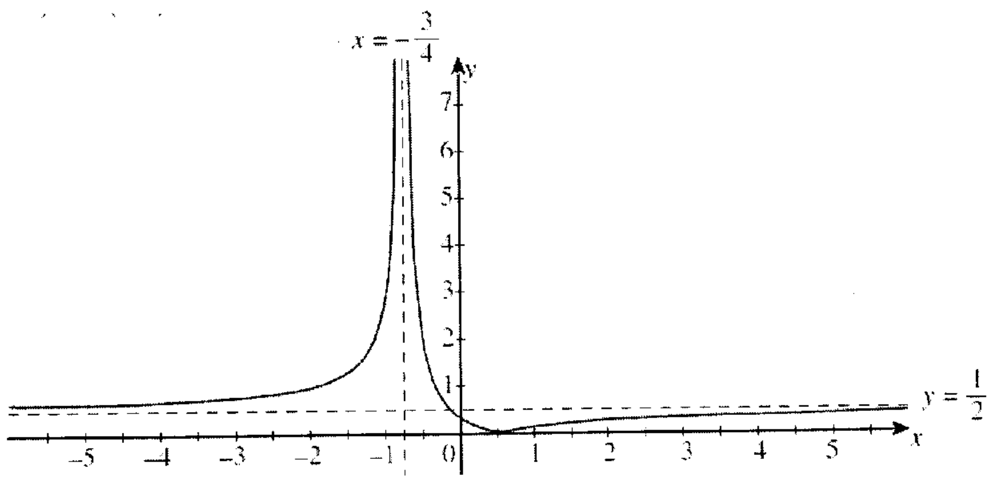
$\therefore$ Intersection point: $(\sqrt{3}, \sqrt{3})$

Total question 5 = 3 marks

PART C: Analysis Question: (10 marks)

A function f, is given by the rule $f(x)=\left|\dfrac{ax+b}{cx+d}\right|$,

Its graph, shown below, has asymptotes with equations $x=-\dfrac{3}{4}$ and $y=\dfrac{1}{2}$. It also passes through the points $\left(\dfrac{1}{2},0\right)$ and $\left(0,\dfrac{1}{3}\right)$



a. State the maximal domain over which f is defined and give its range.

OUTCOME1	OUTCOME3	TOTAL MARKS
2	0	2

Domain: $R\setminus\{-\frac{3}{4}\}$

Range: $[0,\infty)$

b. Given that $f(x)=\frac{1}{2}\left|1-\frac{5}{4x+3}\right|$, find the value of a, b, c and d.

OUTCOME1	OUTCOME3	TOTAL MARKS
2	0	2

$$1-\frac{5}{4x+3}$$

$$=\frac{4x+3}{4x-3}-\frac{5}{4x+3}$$

$$=\frac{4x-2}{4x-3}$$

$$\therefore f(x)=\frac{1}{2}\left|\frac{4x-2}{4x+3}\right|$$

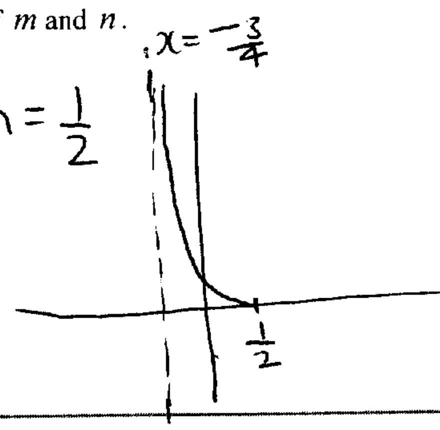
$$=\left|\frac{2x-1}{4x+3}\right|$$

$$\therefore a=2,\ b=-1,\ c=4,\ d=3.$$

c. Consider the function $g:(m,n]\to R,\ g(x)=f(x)$. Given that g^{-1} exists and g has the same range as f.

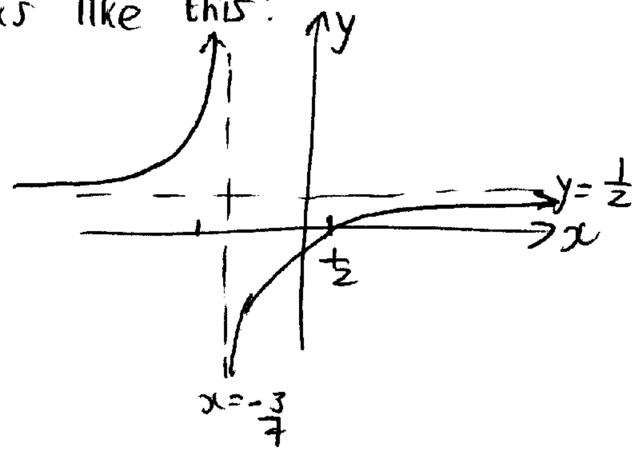
i. State the value of m and n.

$m=\frac{-3}{4},\ n=\frac{1}{2}$



OUTCOME1	OUTCOME3	TOTAL MARKS
1	0	1

If we consider the function $y = \frac{2x-1}{4x+3}$ without the modulus, its graph looks like this:



When we take the modulus, the section of the graph ~~between~~ where $x \in \left(-\frac{3}{4}, \frac{1}{2}\right]$ is reflected in the x-axis.

The equation now becomes: $y = \frac{-(2x-1)}{4x+3} = \frac{1-2x}{4x+3}$.

$$\therefore g(x) = \frac{1-2x}{4x+3} \quad , x \in \left(-\frac{3}{4}, \frac{1}{2}\right]$$

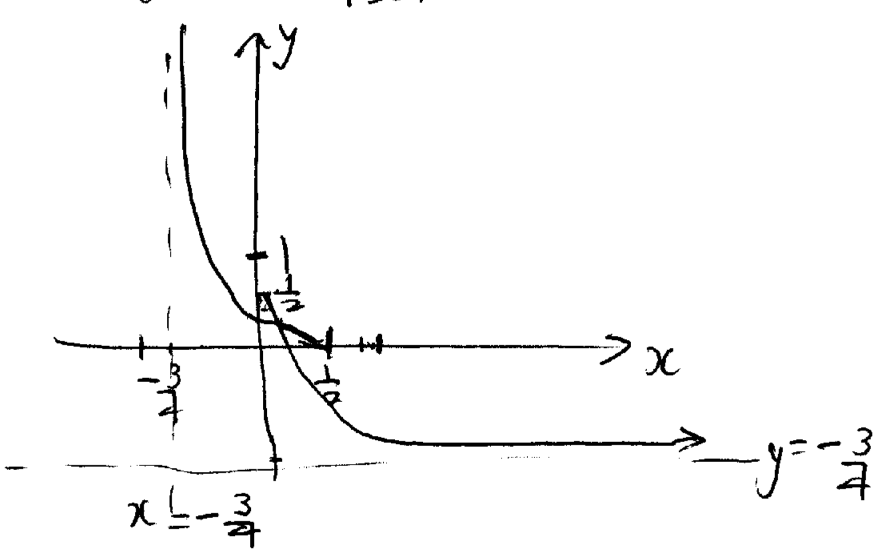
$\text{dom}(g)$	$\text{ran}(g)$
$\left(-\frac{3}{4}, \frac{1}{2}\right]$	$[0, \infty)$
$\text{dom}(g^{-1})$	$\text{ran}(g^{-1})$
$[0, \infty)$	$\left(-\frac{3}{4}, \frac{1}{2}\right]$

$$y = \frac{1-2x}{4x+3}$$

Interchanging: $x = \frac{1-2y}{4y+3}$

$$4y\cancel{x} + 3x = 1-2y$$
$$4yx + 2y = 1-3x$$
$$y = \frac{1-3x}{4x+2}$$

$$\therefore g^{-1}(x) = \frac{1-3x}{4x+2} \quad , x \in [0, \infty)$$



iv. Solve the equation $g^{-1}(x) - g(x) = 0$

Solve : $y = \dfrac{1-2x}{4x+3}$ ①

OUTCOME1	OUTCOME3	TOTAL MARKS
1	0	1

$y = x$ ② where $x \in \left(\frac{-3}{4}, \frac{1}{2}\right]$

Solving : $x = \dfrac{-(\sqrt{41}+5)}{8}$ or $x = \dfrac{\sqrt{41}-5}{8}$

Total of analysis question = 10 marks

END OF THE SCHOOL ASSESSMENT COURSEWORK 1 – MATHS METHODS 3&4 2014

Since $x \in \left(\frac{-3}{4}, \frac{1}{2}\right]$, the required solution is

$$x = \frac{\sqrt{41}-5}{8}$$