SOLUTIONS.

Q 1(a)

$$\begin{bmatrix} 15 \\ 13 \\ 11 \end{bmatrix} \qquad 13 = (15+11) \div 2$$

$\therefore A = 2, D = 13$

(b) $H(t) = 2\sin\left(\frac{2\pi t}{365} + c\right) + 13$

When $t = 171$, $H(171) = 15$

$$\therefore \frac{2\pi \times 171}{365} + c = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$$

All of these give possible values of c.

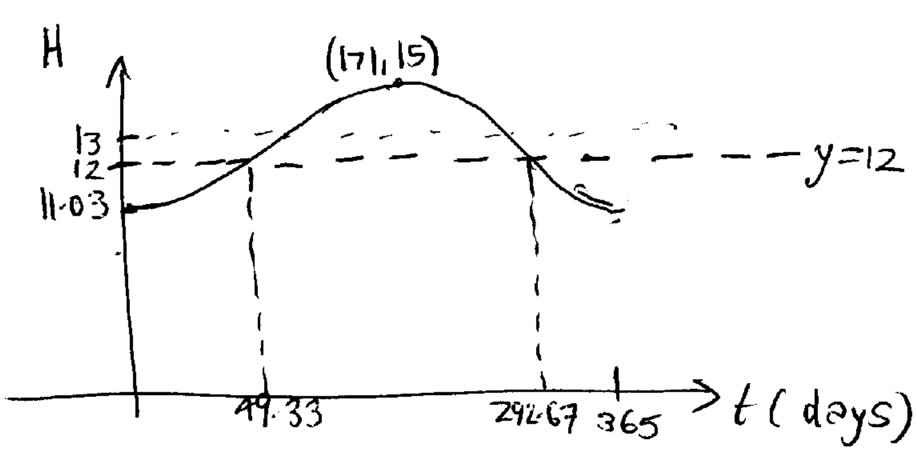
We require the smallest positive value:

$$\frac{2\pi \times 171}{365} + c = \frac{5\pi}{2}$$

$$c = \frac{5\pi}{2} - \frac{2\pi \times 171}{365}$$

$$c = 4.91$$

(c)



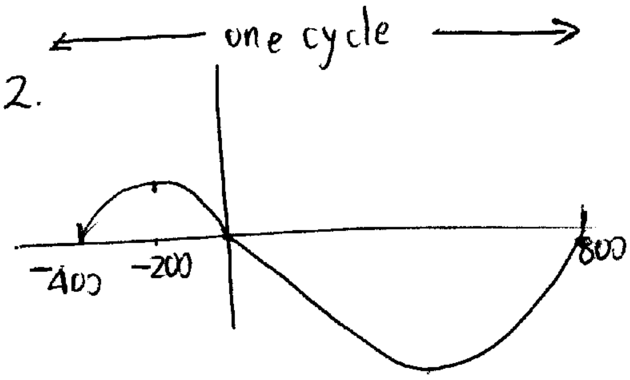
(d) Solving $12 = h(t)$ ~~gives~~ for $0 \le t \le 365$ gives:
$t = 49.33, 292.667$

Q1. [Cont].

$$\text{Required percentage} = \frac{292.667 - 49.333}{365} \times \frac{100}{T}$$

$$= 66.67\%$$

Q2.



We note from the graph that one period is $800 + 400 = 1200$

$$\therefore n = \frac{2\pi}{1200} = \frac{\pi}{600}$$

$$\therefore f(x) = a\cos\left(\frac{\pi}{600}(x - h)\right) + k$$

~~Ef~~ We also note that the graph represents a cosine function translated 200 units to the left.

$$\therefore h = -200$$

$$f(x) = a\cos\left(\frac{\pi}{600}(x + 200)\right) + k$$

$f(\overset{0}{\cancel{400}}) = 0$ $\quad\therefore 0 = a\cos\left(\frac{\pi}{3}\right) + k$

$$\therefore 0 = \frac{a}{2} + k \quad \therefore k = -\frac{a}{2}$$

$$\therefore f(x) = a\cos\left(\frac{\pi}{600}(x + 200)\right) - \frac{a}{2}$$

$f(100) = -90$ $\quad\therefore -90 = a\cos\left(\frac{300\pi}{600}\right) - \frac{a}{2}$

$$\therefore -90 = a \times 0 - \frac{a}{2} \qquad \therefore a = 180.$$

∴ Equation of rock stratum:

$$f(x) = 180\cos\left(\frac{\pi}{600}(x+200)\right) - 90$$

(b) $f(510) = -240.96$

$\therefore \approx 241$ m depth

(c) (i)

$$\left[\begin{array}{l} 90 \\ -90 \\ -270 \end{array}\right.$$

Maximum depth = 270 m

(ii) $-270 = 180\cos\left(\frac{\pi}{600}(x+200)\right) - 90$

$$\therefore -1 = \cos\left(\frac{\pi}{600}(x+200)\right)$$

$$\frac{\pi}{600}(x+200) = -\pi \pm 2n\pi$$

$$\frac{1}{600}(x+200) = (2n-1)$$

$$x = (2n-1) \times 600 - 200$$

If $n = 1$: $x = 1 \times 600 - 200 = 400$

If $n = 2$: $x = 3 \times 600 - 200 = 1600$

If $n = 3$: $x = 5 \times 600 - 200 = 2800$

If $n = 4$: $x = 7 \times 600 - 200 = 4000$

If $n = 5$: $x = 9 \times 600 - 200 = 5200$.

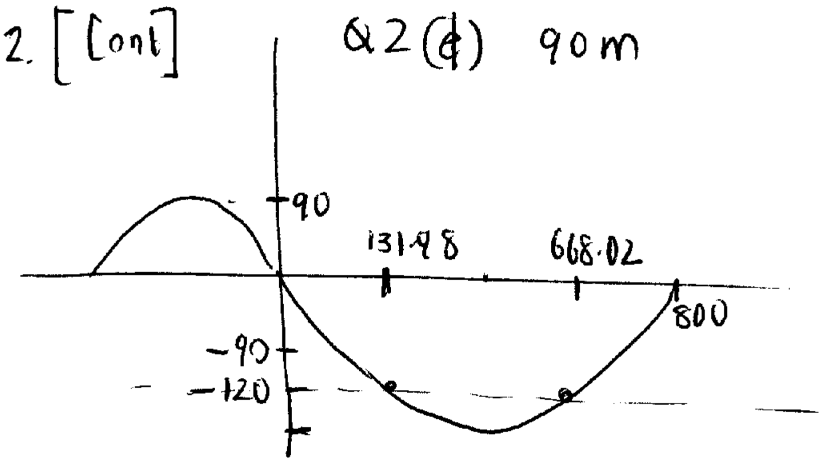
For $x : x \in [0, 5000]$, the values of x for which depth is a maximum are:

$x = 400, 1600, 2800, 4000$.

3

Q2. [Cont]

Q2(d) 90m



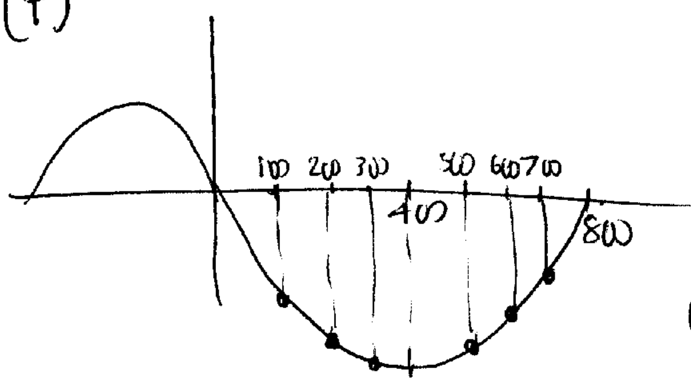
Q2(e) Solving: $f(x) = -120$ for $0 \leq x \leq 800$

gives: $x = 131.98, 668.02$

Rock stratum is within 120 m of surface for

$\{x : 0 \leq x \leq 132 \cup 668 \leq x \leq 800.\}$

Q2(f)



Cost of a hole
= length of hole × 75
= $75|y|$ where y is the y-co-ordinate of each endpoint of hole.

We note: $f(700) = f(100)$
$f(200) = f(600)$
$f(300) = f(500)$

The total cost will therefore be:

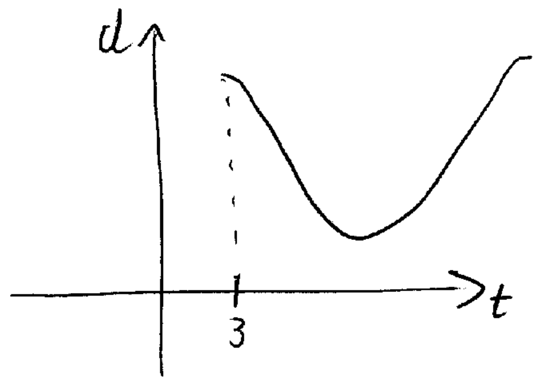
$$\left| 2 \times f(100) \times 75 + 2 \times f(200) \times 75 + 2 \times f(300) \times 75 + 75 \times f(400) \right|$$

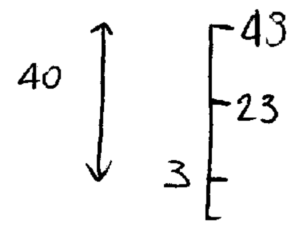
$$= \cancel{150} \; 75 \left| 2f(100) + 2f(200) + 2f(300) + f(400) \right|$$

Having defined $f(x)$, this calculation is quite quick:
= \$ 97,633.

Q3.(a)

At $t=3$, your vertical distance d is a maximum





$$\frac{(43+3)}{2} = 23$$

So this represents a cosine function translated 3 units to the right.

From above, $a=20$, $c=23$

$$\therefore \quad d(t) = 20\cos\left(n(t-3)\right)+23$$

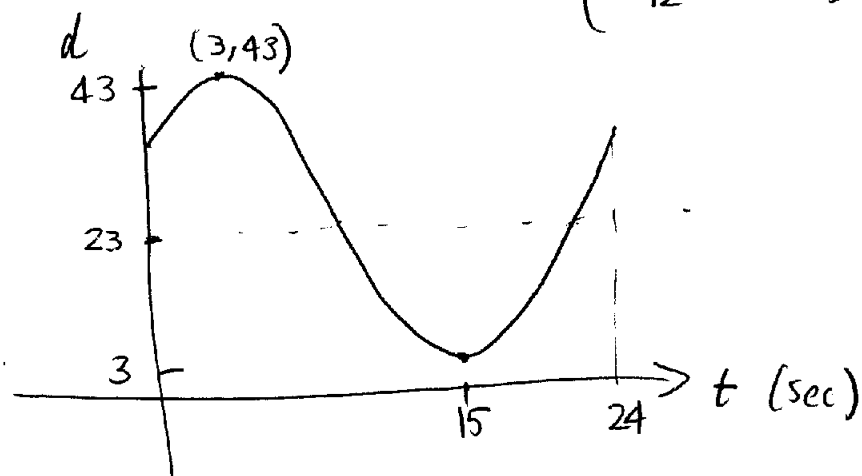
Since one complete cycle takes 24 seconds, the period $=24$

$$\therefore \frac{2\pi}{n} = 24$$

$$\therefore n = \frac{\pi}{12}$$

$$\therefore d(t) = 20\cos\left(\frac{\pi}{12}(t-3)\right)+23.$$

(b)



(c) $d(20) = 17.82$ ft.

(d

Q3 [Cont]

Solving $18 = 20\cos\left(\frac{\pi}{12}(t-3)\right)+23$: for $0 \le t \le 24$ gives: $t = 9.97,\ 20.03$

Since the first three times are required, the third time will be $24 + 9.97 = 33.97$

$\therefore$ Required times:

$t = 9.97,\ 20.03,\ 33.97$

Q4. (a)

$10+9.8 = 19.8$

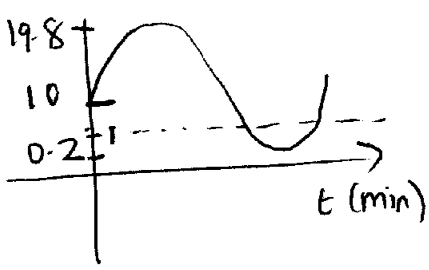
10

$10-9.8 = 0.2$

Max depth $= 19.8$ m

Min depth $= 0.2$ m

(b) At $t=0$, the height of water is 10m so we have a sine function.



Period $= 15$ min

$\therefore \frac{2\pi}{n} = 15$

$\therefore n = \frac{2\pi}{15}$

$$h(t) = 9.8\sin\left(\frac{2\pi t}{15}\right) + 10$$

(c) Solving $h(t) = 1$ for $0 \le t \le 15$ gives:

$t = 10.28,\ 12.22$

Between $t = 10.28$ and $t = 12.22$, the height is less than 1 m.

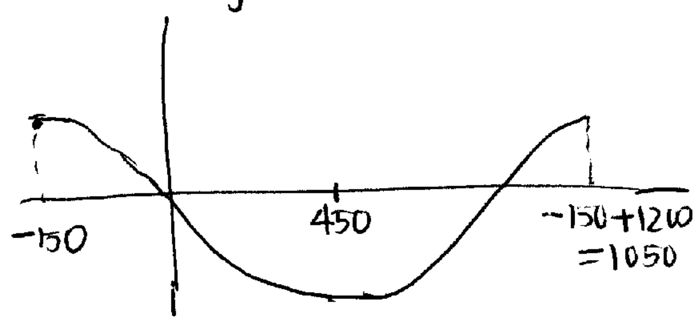
Q 4 (d) Travelling at 1200 km/hr, since the period is 15 min
the distance between successive crests $= 1200 \div 4$

$= 300$ m.

Q 5. (a)

$-70 \left[\begin{array}{l} -70+100 = 30 \\ -70-100 = -170 \end{array} \right.$

Maximum height of island $= 30$ m.



$y = -70 + 100\cos\left(\frac{\pi(x+150)}{600}\right)$ Period $= \frac{2\pi}{\left(\frac{\pi}{600}\right)} = 1200$

∴ Max height occurs at $x = 1050$.

(b) Greatest depth $= 170$ m when $x = 450$.

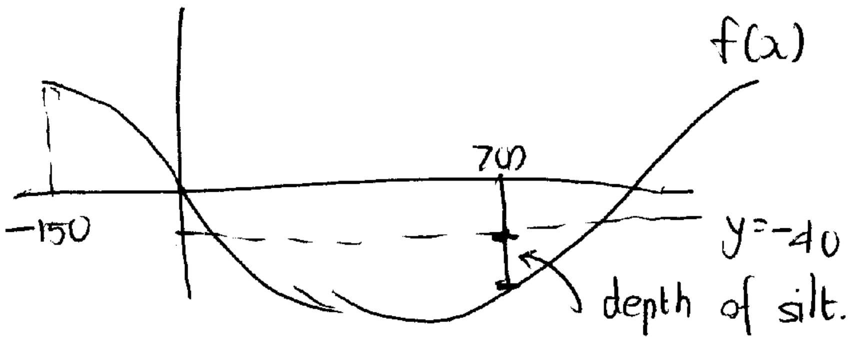
(c) Solving $-40 = -70 + \cancel{80}\ 100\cos\left(\frac{\pi(x+150)}{600}\right)$

for $0 \le x \le 1050$ gives:

$x = 91.08,\ 808.19$

∴ For $\{x : x \in [91, 808]\}$ he will find silt.

Q5. (d)



The depth of silt will be: $-40 - f(700)$

$= 55.88$ m

He will have to dig through 55.88 m of silt.

(e)

$f(0) = -70 + 100\cos\left(\frac{150\pi}{600}\right)$

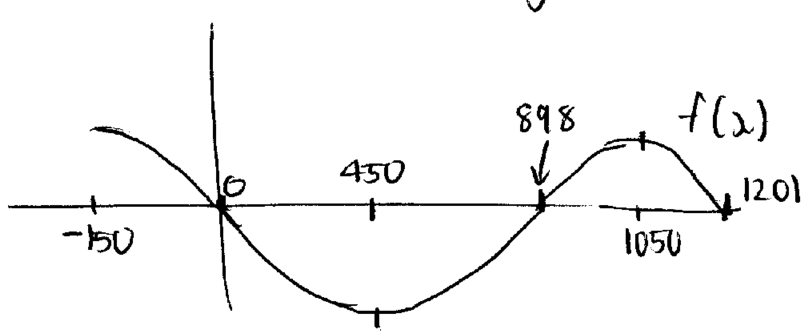
$= -70 + 100\cos\left(\frac{\pi}{4}\right)$

$= \frac{100\sqrt{2}}{2} - 70 \quad = 50\sqrt{2} - 70$

≈ 0.71

It just misses the origin.

(f)



Solving $f(x) = 0$ for $0 \leq x \leq 1500$ gives:

$x = 1.91, 898.09, 1201.91$

$\therefore$ Width of island $= 1201.91 - 898.09$

≈ 304 m.

Q6(a) $(1948 - 1750)/18 = 11$

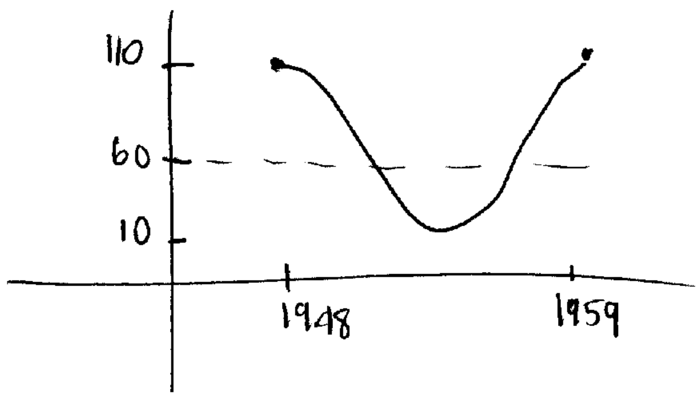
Period = 11 years $\therefore \frac{2\pi}{n} = 11 \therefore n = \frac{2\pi}{11}$

(b) $\begin{bmatrix} 110 \\ 60 \\ 10 \end{bmatrix}$ $\uparrow +50$ $\downarrow -50$

Amplitude $= 50$

Vertical shift $= 60$.

Maximum occured at $t = 1948$



This graph could have been obtained by shifting the cosine function $y = 50\cos\left(\frac{2\pi t}{11}\right) + 60$ 1948 units to the right.

$$\therefore N(t) = 50\cos\left(\frac{2\pi(t-1948)}{11}\right) + 60.$$

(c) Defining $N(t) = 50\cos\left(\frac{2\pi(t-1948)}{11}\right) + 60,$

$$N(2014) = 110.$$

(2014 is a year of maximum sunspot activity!)

(d) $N(2020) = 12.025$

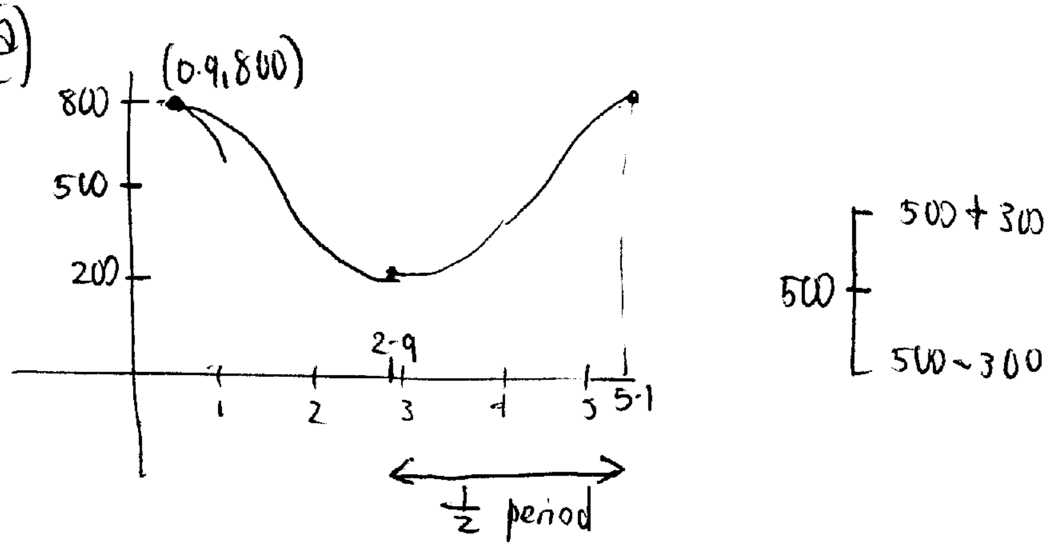
$\simeq 12$ sunspots

(e) solving $n(t) = 53$ for $2014 \leq t \leq 2025$

gives: $t = 2017$ or 2022

$2017 - 2014 = 3$. $\therefore$ In 3 years from now.

Q7. (a)



$\frac{1}{2}$ period $= 5.1 - 2.9$
$= 2.1$
$\therefore$ Period $= 4.2$ $\quad \frac{2\pi}{n} = 4.2 \therefore n = \frac{2\pi}{4.2}$

$\therefore$ First maximum occurred at $t = 5.1 - 4.2 = 0.9$

This represents a cosine function shifted 0.9 units to right.

$$\therefore \quad P(t) = 300 \cos\left(\frac{2\pi}{4.2}(t - 0.9)\right) + 500$$

$$\therefore P(t) = 300 \cos\left(\frac{\pi}{2.1}(t - 0.9)\right) + 500$$

$\therefore$ $A = 300$, $C = 500$, $b = 2.1$, $k = 0.9$

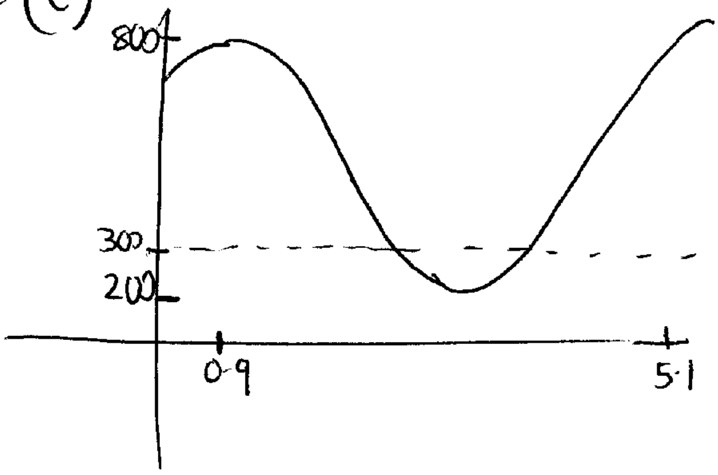
(b) (i) Defining $P(t) = 300 \cos\left(\frac{\pi}{2.1}(t - 0.9)\right) + 500$,

$P(7) = 213.3$ $\therefore$ Population about 213

(ii) $P(8) = 390.4$ $\therefore$ Population about 390.

(iii) $P(9) = 770.29$ Population about 770.

Q7. (c)

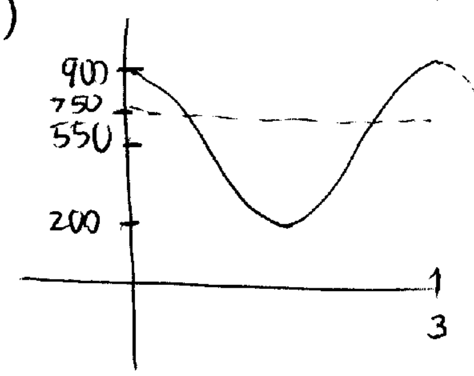


Solving $P(t) = 300$ for $0.9 \leq t \leq 5.1$
gives $t = 2.44,\ 3.56$.

Between $t = 2.44$ and $t = 3.56$ the population was declared vulnerable.

Q 8(d)

Basic cosine shape



+350 ↑ 900
550
−350 ↓ 200

$(900+200) \div 2$
$= 550, \; a = 350$

$\frac{2\pi}{n} = 3$
$\therefore n = \frac{2\pi}{3}$

$R(t) = 550 + 350\cos\left(\frac{2\pi t}{3}\right)$

(b) $R(7) = 550 + 350\cos\left(\frac{14\pi}{3}\right)$
$= 725.$

$\therefore$ Count will be 725.

(c) Solving $750 = R(t)$ for $0 \le t \le 3$ gives
$t = 0.4596, \; 2.5404$

$\therefore$ Longest continuous interval over the period of treatment where $R(t) \geq 750$ will be: 0.4596×2
$= 0.9192$ weeks.

(d) Overall percentage $= \frac{0.9192}{3} \times \frac{100}{1}$
$= 30.64\%$

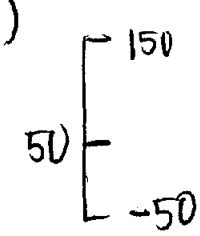
Q9

$$y = 100\cos\left(\frac{\pi(x-400)}{600}\right) + 50$$

$$\text{Period} = \frac{2\pi}{\frac{\pi}{600}} = 1200.$$

The graph is a cosine function shifted 400 units to the right, so the first maximum occurs at $x = 400$.

(a)



Max height = 150 m

(b) Lowest point of valley is 50 m below bridge.

(c) Defining $f(x) = 100\cos\left(\frac{\pi(x-400)}{600}\right) + 50$,

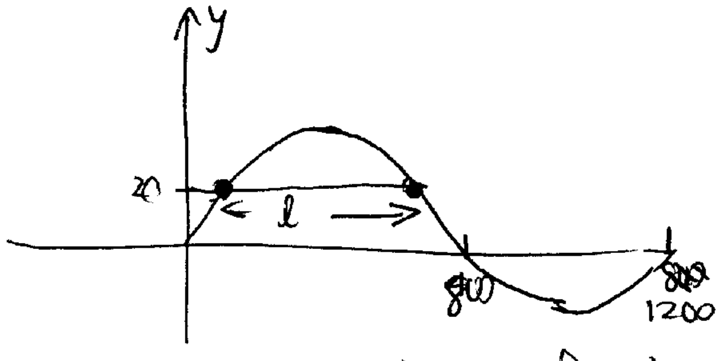
solve $f(x) = 0$ for $0 \le x \le 1200$ gives:

$x = 0, 800, 1200.$

(i) Length of tunnel = 800m

(ii) Length of bridge = 400m

(d)

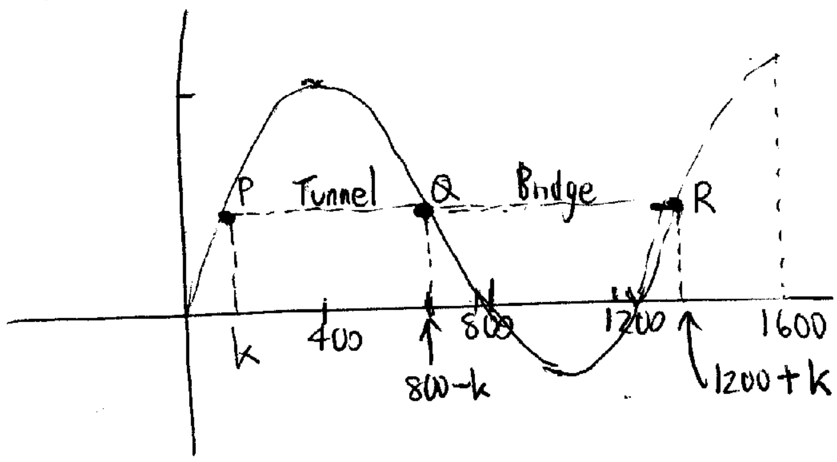


Solving $f(x) = 20$ for $0 \le x \le 800$

gives: $x = 41.808, 758.192$

$\therefore$ Length $\simeq 716.38 \simeq 716$ m.

Q9 (e)



Note that point R is one period away from P.
Therefore, the x-co-ordinate of R is $1200+k$.
By symmetry, the x-coordinate of Q is $800-k$

(i) Length of tunnel $= (800-k)-k$
$= 800-2k$

(ii) Length of bridge $= (1200+k)-(800-k)$
$= 400+2k$

(f) $C = (800-2k)^2 + (400+2k)^2$

(g) $C = (800)^2 - 3200k + 4k^2 + (400)^2 + 1600k + 4k^2$
$C = 8k^2 - 1600k + 640000 + 160000$
$C = 8(k^2-200k) + 800000$
$C = 8(k^2-200k+(100)^2) + 800000 - 80000$
$C = 8(k-100)^2 + 720{,}000$

∴ Minimum cost occurs when $k=100$.