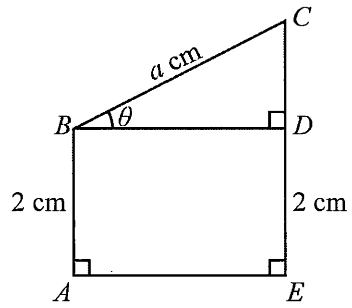
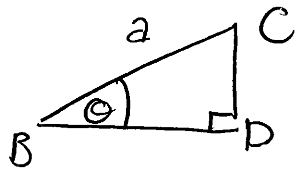
Question 10

The figure shown represents a wire frame where $ABCE$ is a convex quadrilateral. The point D is on line segment EC with $AB = ED = 2$ cm and $BC = a$ cm, where a is a positive constant.

$$\angle BAE = \angle CEA = \frac{\pi}{2}$$

Let $\angle CBD = \theta$ where $0 < \theta < \frac{\pi}{2}$.





$\cos\theta = \frac{BD}{a}$

$\sin\theta = \frac{CD}{a}$

a. Find BD and CD in terms of a and θ.

$BD = a\cos\theta$

$CD = a\sin\theta$

2 marks

b. Find the length, L cm, of the wire in the frame, including length BD, in terms of a and θ.

$L = a + 2 + AE + 2 + CD + BD$

$\therefore L = a + 4 + a\cos\theta + a\sin\theta + a\cos\theta$

$L = a + 4 + 2a\cos\theta + a\sin\theta$

1 mark

c. Find $\frac{dL}{d\theta}$, and **hence** show that $\frac{dL}{d\theta} = 0$ when $BD = 2CD$.

$$\frac{dL}{d\theta} = -2a\sin\theta + a\cos\theta$$

Let $\frac{dL}{d\theta} = 0$

$\therefore \quad a\cos\theta - 2a\sin\theta = 0$

But $\overline{BD} = a\cos\theta$ and $\overline{CD} = a\sin\theta$

$\therefore \quad \frac{dL}{d\theta} = 0$ when $BD = 2CD$

2 marks

d. Find the maximum value of L if $a = 3\sqrt{5}$. If $a = 3\sqrt{5}$

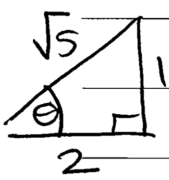
$L = 3\sqrt{5} + 4 + 6\sqrt{5}\cos\theta + 3\sqrt{5}\sin\theta$.

For a maximum, $\frac{dL}{d\theta} = 0$

$\therefore \quad a\cos\theta - 2a\sin\theta = 0$

$\therefore \quad \cos\theta = 2\sin\theta$

$\therefore \quad \frac{1}{2} = \tan\theta$.



If $\tan\theta = \frac{1}{2}$, then: $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$

$\therefore$ When $\tan\theta = \frac{1}{2}$,

$$L = 3\sqrt{5} + 4 + 6\sqrt{5} \times \frac{2}{\sqrt{5}} + 3\sqrt{5} \times \frac{1}{\sqrt{5}}$$

1 mark

$$= 3\sqrt{5} + 4 + 12 + 3$$

$$= 19 + 3\sqrt{5}$$

END OF QUESTION AND ANSWER BOOK

Question 10

Let $f: R \to R$, $f(x) = e^{-mx} + 3x$, where m is a positive rational number.

a. **i.** Find, in terms of m, the x-coordinate of the stationary point of the graph of $y = f(x)$.

$f'(x) = -me^{-mx} + 3$

For a stationary point, $f'(x) = 0$

$\therefore \quad 3 - me^{-mx} = 0$

$\therefore \quad \frac{3}{m} = e^{-mx}$

$-mx = \log_e\left(\frac{3}{m}\right) \quad \therefore \quad x = -\frac{1}{m}\log_e\left(\frac{3}{m}\right)$

ii. State the values of m such that the x-coordinate of this stationary point is a positive number.

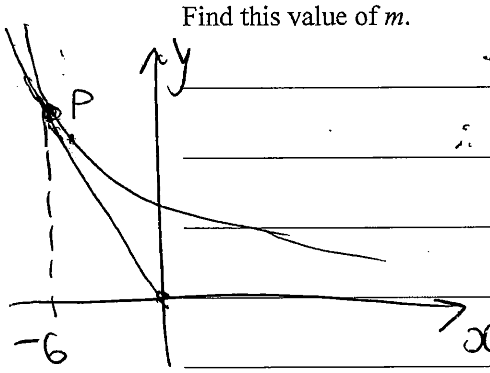
If $m > 0$, $-\frac{1}{m} < 0$. So x will only be positive if $\log_e\left(\frac{3}{m}\right) < 0$

$\therefore \quad \frac{3}{m} < 1 \quad \therefore \quad m > 3$

2 + 1 = 3 marks

b. For a particular value of m, the tangent to the graph of $y = f(x)$ at $x = -6$ passes through the origin.

Find this value of m.



$f(x) = e^{-mx} + 3x$

$\therefore f(-6) = e^{6m} - 18.$

$\therefore P = (-6, e^{6m} - 18).$

$f'(x) = -me^{-mx} + 3$

$f'(-6) = -me^{6m} + 3.$

$\therefore \quad -me^{6m} + 3 = \frac{e^{6m} - 18 - 0}{-6 - 0}$

$\therefore \quad 3 - me^{6m} = \frac{e^{6m} - 18}{-6}$

3 marks

$\therefore -18 + 6me^{6m} = e^{6m} - 18$

$\therefore \quad 6me^{6m} = e^{6m}$

$\therefore \quad 6m = 1$

$m = \frac{1}{6}$

END OF QUESTION AND ANSWER BOOK