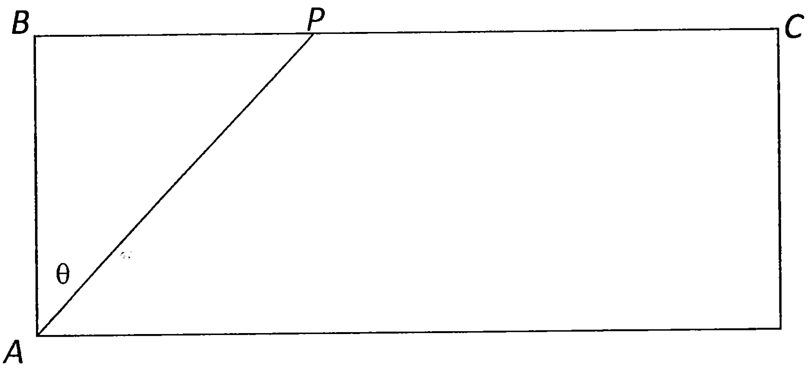
Class Question : 13 July

The figure below shows a rectangular field in which AB = 300 m and BC = 1100 m.

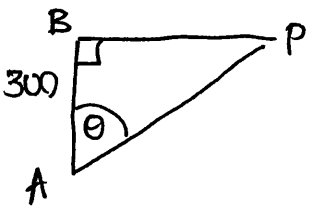


AB = 300 m

BC = 1100 m

Tasmania Jones is an athlete, and he needs to get from A to C as rapidly as possible. He can run across the rectangular field at 4 m/sec. There is a path along BC, and he can run along the path faster than across the grassy field. He can run along the path at 5 m/sec.

a. Tasmania Jones runs across the field from A to P at 4 m/sec. He runs in a direction that makes an angle θ with AB, as shown above. Find the time he takes to run from A to P in terms of θ.



$\cos\theta = \frac{\cancel{A}\,300}{AP}$

$AP = \frac{300}{\cos\theta}$

$\therefore \text{Time} = \frac{300}{\cos\theta} \div 4$

$= \frac{75}{\cos\theta}$

b. Find the time that Tasmania takes to run from P to C along the path in terms of θ.

$\tan(\theta) = \frac{PB}{300}$

$\overline{PB} = 300\tan\theta$

$\therefore \quad PC = 1100 - 300\tan\theta$

$\text{Time} = \frac{1100 - 300\tan\theta}{5}$

$= 220 - 60\tan\theta$

c. Show, using a and b above, that the total time T seconds for running from A to C is given by:

$$T(\theta) = 220 + \frac{75 - 60\sin(\theta)}{\cos(\theta)}$$

$T(\theta) = \frac{75}{\cos\theta} + 220 - 60\tan\theta = \frac{75}{\cos\theta} + 220 - \frac{60\sin\theta}{\cos\theta}$

$= 220 + \frac{75 - 60\sin\theta}{\cos\theta}$

d. Find $\frac{dT}{d\theta}$.

$T(\theta) = 75(\cos\theta)^{-1} + 220 - 60\tan(\theta)$

$T'(\theta) = 75 \times -1\,(\cos\theta)^{-2} \times -\sin\theta - 60\sec^2\theta$

$= \frac{75\sin\theta}{\cos^2\theta} - \frac{60}{\cos^2\theta}$

e. Find the value of θ for which the time taken will be a minimum. Justify that this value of θ will give a minimum

Let $\frac{dT}{d\theta} = 0$ for a stationary point

$\therefore \frac{75\sin\theta}{\cos^2\theta} - \frac{60}{\cos^2\theta} = 0$

0.927

$75\sin\theta = 60$

$\therefore \sin(\theta) = \frac{60}{75} = \frac{4}{5}$

$\therefore \theta = \sin^{-1}\left(\frac{4}{5}\right)$

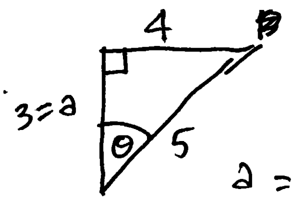
$\theta = 0.9273^c$

$T'(0.8) \approx -13$

$T'(1) \approx 11$

$\therefore$ Local minimum t/p.

f. Find the distance of point P from B, where P is the point to which Tasmania should run to minimize his total time. Find also the minimum total time

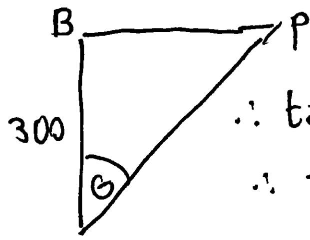


For a minimum

$\sin(\theta) = \frac{4}{5}$

$a = \sqrt{5^2 - 4^2} = 3$

$\therefore \tan(\theta) = \frac{4}{3}$



$\therefore \tan(\theta) = \frac{PB}{300}$

$\therefore \frac{4}{3} = \frac{PB}{300}$

$PB = 400$

He should run to 400m from B.

$T\left(\sin^{-1}\left(\frac{4}{5}\right)\right) = 265$

Minimum time $= 265$ sec.

g. If instead, Tasmania can run across the field at u m/sec (instead of 4 m/sec), but can still run along the path BC at 5 m/sec, find the values of u for which it would be fastest for Tasmania to simply run directly from A to C. Give your answer correct to two decimal places.

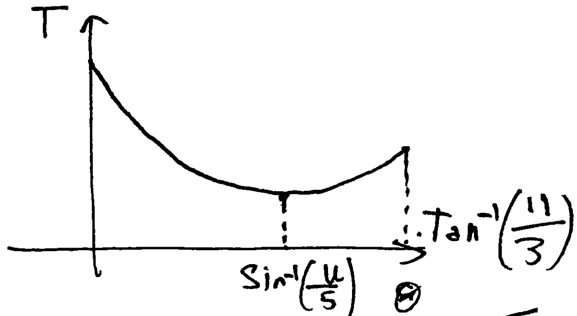
$T(\theta) = \frac{300}{u\cos\theta} + 220 - 60\tan\theta$

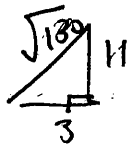
$\therefore T'(\theta) = \frac{300\sin\theta}{u\cos^2\theta} - \frac{60}{\cos^2\theta}$

$T'(\theta) = 0$ if $\frac{300\sin\theta}{u} = 60$

$\therefore \sin\theta = \frac{60u}{300}$

$= \frac{u}{5}$





$\therefore \frac{u}{5} \geq \frac{11}{\sqrt{130}}$

For it to be best to run directly,

$\sin^{-1}\left(\frac{u}{5}\right) \geq \tan^{-1}\left(\frac{11}{3}\right)$

$u \geq \frac{55}{\sqrt{130}}$

$u \geq 4.82$