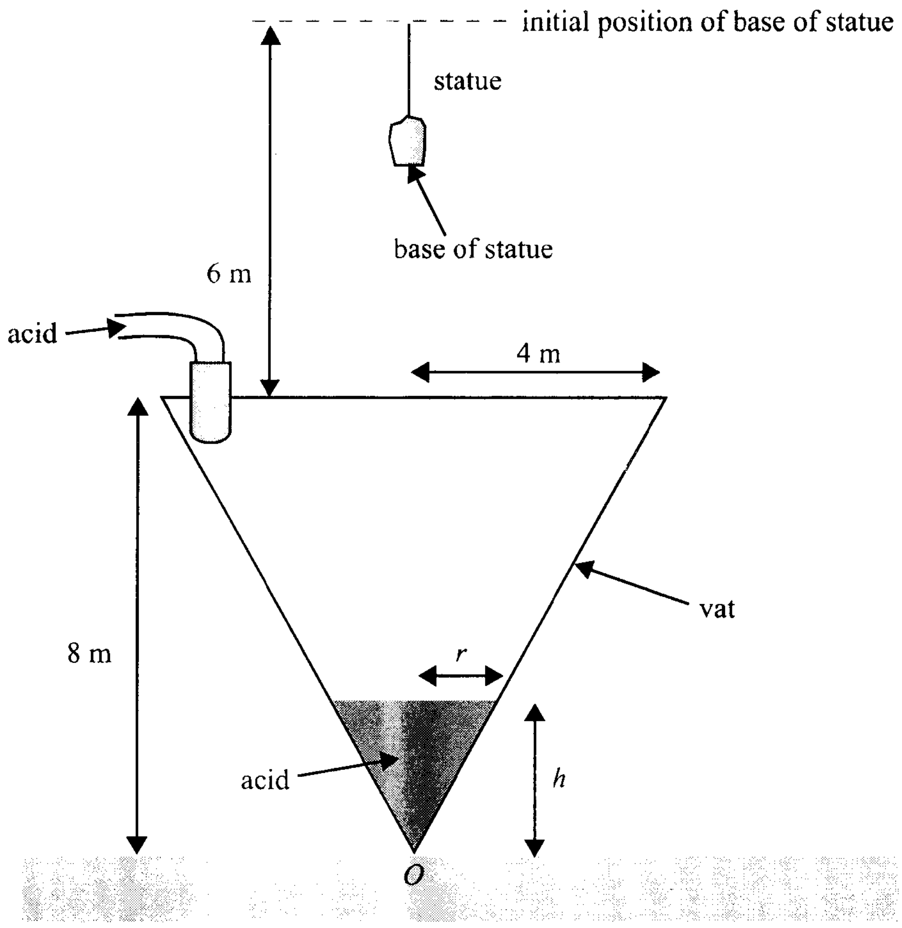
Question 4

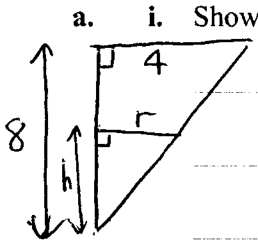
A Zambeji tribe has stolen a precious marble statue from Tasmania Jones. The statue has been tied to a rope and is suspended so that its base is initially 6 metres above the top of a vat. The vat is an inverted right circular cone with base radius 4 metres and height 8 metres.



At 9.00 am the tribe starts to lower the marble statue towards the vat at a rate of 1 metre per hour. At the same time acid begins to be poured into the vat at a constant rate of $\frac{9\pi}{4}$ m^3 per hour. The vat is initially empty. When the statue touches the acid, it will start to dissolve.

At time t hours after 9.00 am, the height of acid in the vat is h metres and the radius of the surface of the acid in the vat is r metres.

a. i. Show that $h = 2r$.



Similar triangles: $\frac{r}{4} = \frac{h}{8}$

$\therefore h = 2r$

ii. Hence find an expression for the volume of acid in the vat at time t, in terms of h.

$V = \frac{\pi r^2 h}{3}$

$h = 2r$

$\therefore r = \frac{h}{2}$

$\therefore V = \frac{\pi h}{3} \times \left(\frac{h}{2}\right)^2$

$\therefore V = \frac{\pi h^3}{12}$

1 + 1 = 2 marks

b. Show that $\frac{dh}{dt}$, the rate at which the height of the acid is increasing, is $\frac{9}{h^2}$ metres per hour.

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

Given: $\frac{dV}{dt} = \frac{9\pi}{4}$

$V = \frac{\pi h^3}{12}$ $\therefore \frac{dV}{dh} = \frac{\pi h^2}{4}$

$\therefore \frac{dh}{dV} = \frac{4}{\pi h^2}$

$\therefore \frac{dh}{dt} = \frac{4}{\pi h^2} \times \frac{9\pi}{4} = \frac{9}{h^2}$, as stated.

2 marks

c. Find, giving exact values

i. the rate at which the height of the acid is increasing when its height is 2 metres

$\frac{dh}{dt} = \frac{9}{h^2}$

When $h=2$, $\frac{dh}{dt} = \frac{9}{2^2} = \frac{9}{4}$

ii. the height of the acid when it is increasing at half the rate found in **c. i.**

When $\frac{dh}{dt} = \frac{9}{8}$

$\frac{9}{h^2} = \frac{9}{8}$

$\therefore h^2 = 8$

$h = \sqrt{8} = 2\sqrt{2}$

1 + 1 = 2 marks

d. **i.** Write an expression for $\frac{dt}{dh}$ in terms of h.

$\frac{dh}{dt} = \frac{9}{h^2}$ $\therefore$ $\frac{dt}{dh} = \frac{h^2}{9}$

ii. Hence find an expression for the height of the acid in terms of t.

$t = \int \frac{h^2}{9}\, dh$

$\therefore t = \frac{h^3}{27} + c$

When $t = 0,\ h = 0$ $\therefore$ $t = \frac{h^3}{27}$

$\therefore h^3 = 27t$

$h = (27t)^{1/3}$ $\therefore h = 3t^{1/3}$

1 + 1 = 2 marks

Tasmania Jones will try to save the statue.

e. **i.** Write an expression for the distance of the base of the statue above ground level t hours after 9.00 am. (The vertex of the cone, O, is at ground level.)

$14 - t$

ii. At what time would the statue first touch the acid?

$3t^{1/3} = 14 - t$

$\therefore t = 8$

$\therefore$ At 8 hours after 9:00AM

= 5:00PM.

1 + 2 = 3 marks

Total 11 marks

END OF QUESTION AND ANSWER BOOK

Question 3

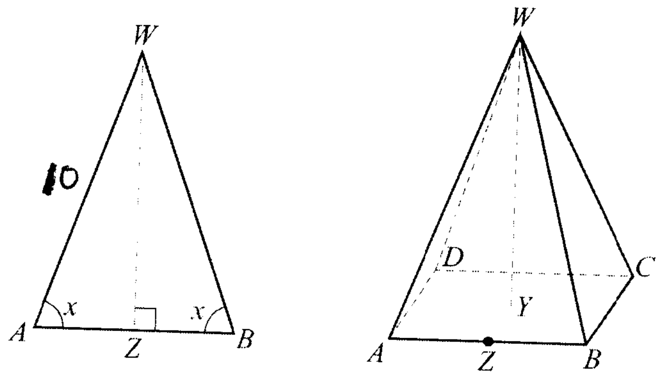
An ancient civilisation buried its kings and queens in tombs in the shape of a square-based pyramid, *WABCD*.

The kings and queens were each buried in a pyramid with $WA = WB = WC = WD = 10$ m.

Each of the isosceles triangle faces is congruent to each of the other triangular faces.

The base angle of each of these triangles is x, where $\frac{\pi}{4} < x < \frac{\pi}{2}$.

Pyramid *WABCD* and a face of the pyramid, *WAB*, are shown here.



Z is the midpoint of *AB*.

a. **i.** Find *AB* in terms of x.

$\cos(x) = \frac{AZ}{10}$ $\therefore AZ = 10\cos(x)$

$\therefore \overline{AB} = 20\cos(x)$

ii. Find *WZ* in terms of x.

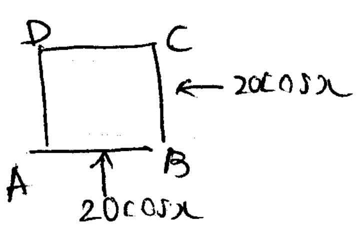
$\sin(x) = \frac{WZ}{10}$

$\therefore WZ = 10\sin(x)$

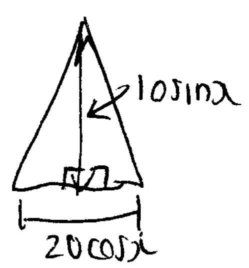
1 + 1 = 2 marks

b. Show that the total surface area (including the base), S m^2, of the pyramid, *WABCD*, is given by $S = 400(\cos^2(x) + \cos(x)\sin(x))$.

$S = 4 \times A_{\text{triangle}} + A_{\text{square}}$



$A_{\text{square}} = (20\cos(x))^2 = 400\cos^2 x$



$A_{\text{triangle}} = \frac{1}{2} \times 20\cos x \times 10\sin x$

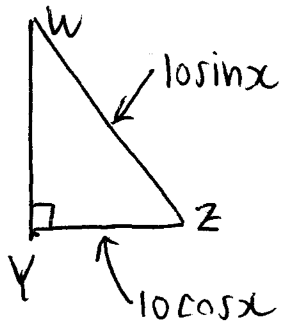
$= 100\cos x\sin x$

2 marks

$\therefore S = 4 \times 100\cos x\sin x + 400\cos^2 x$

$= 400(\cos^2 x + \cos x\sin x)$

c. Find *WY*, the height of the pyramid *WABCD*, in terms of *x*.



$$\overline{WY}^2 = (10\sin(x))^2 - (10\cos(x))^2$$

$$= 100(\sin^2 x - \cos^2 x)$$

$$\therefore \overline{WY} = \sqrt{100(\sin^2 x - \cos^2 x)}$$

$$= 10\sqrt{\sin^2 x - \cos^2 x}$$

2 marks

d. The volume of any pyramid is given by the formula Volume $= \frac{1}{3} \times$ area of base $\times$ vertical height.

Show that the volume, T m^3, of the pyramid *WABCD* is $\frac{4000}{3}\sqrt{\cos^4 x - 2\cos^6 x}$.

$$T = \frac{1}{3} A_{square} \times \overline{WY}$$

$$= \frac{1}{3} \times 400\cos^2 x \times 10\sqrt{\sin^2 x - \cos^2 x}$$

$$= \frac{4000}{3} \cos^2 x \sqrt{1 - \cos^2 x - \cos^2 x}$$

$$= \frac{4000}{3} \cos^2 x \sqrt{1 - 2\cos^2 x} = \frac{4000}{3}\sqrt{\cos^4 x - 2\cos^6 x}$$

1 mark

Queen Hepzabah's pyramid was designed so that it had the **maximum possible volume**.

e. Find $\frac{dT}{dx}$ and **hence** find the exact volume of Queen Hepzabah's pyramid and the corresponding value of x.

$$T = \frac{4000}{3} \cos^2 x \sqrt{\cos^4 x - 2\cos^6 x}$$

$$\frac{dT}{dx} = \frac{8000 \sin x (\cos x)^3}{3\sqrt{1 - 2\cos^2 x}} - \frac{8000 \sin x \cos x \sqrt{1 - 2\cos^2 x}}{3}$$

$= 0$ for a stationary point.

$$\therefore \frac{8000 \sin x \cos x}{3}\left(\frac{\cos^2 x}{\sqrt{1 - 2\cos^2 x}} - \sqrt{1 - 2\cos^2 x}\right) = 0$$

where $\frac{\pi}{4} < x < \frac{\pi}{2}$

4 marks

$$\therefore \frac{\cos^2 x}{\sqrt{1 - 2\cos^2 x}} = \sqrt{1 - 2\cos^2 x} \quad \therefore 3\cos^2 x = 1 \quad \therefore \cos x = \frac{1}{\sqrt{3}}$$

$\therefore$ Max volume $= \frac{4000\sqrt{3}}{27}$

$\therefore x = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $T_{max} = T\left(\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{4000\sqrt{3}}{27}$

Queen Hepzabah's daughter, Queen Jepzibah, was also buried in a pyramid. It also had

$$WA = WB = WC = WD = 10 \text{ m}.$$

The volume of Jepzibah's pyramid is exactly one half of the volume of Queen Hepzabah's pyramid. The volume of Queen Jepzibah's pyramid is also given by the formula for T obtained in **part d.**

f. Find the possible values of x, for Jepzibah's pyramid, correct to two decimal places.

$$\frac{2000\sqrt{3}}{27} = \frac{4000}{3}\sqrt{\cos^4 x - 2\cos^6 x}, \quad \frac{\pi}{4} < x < \frac{\pi}{2}$$

$$x = 0.81,\ 1.23$$

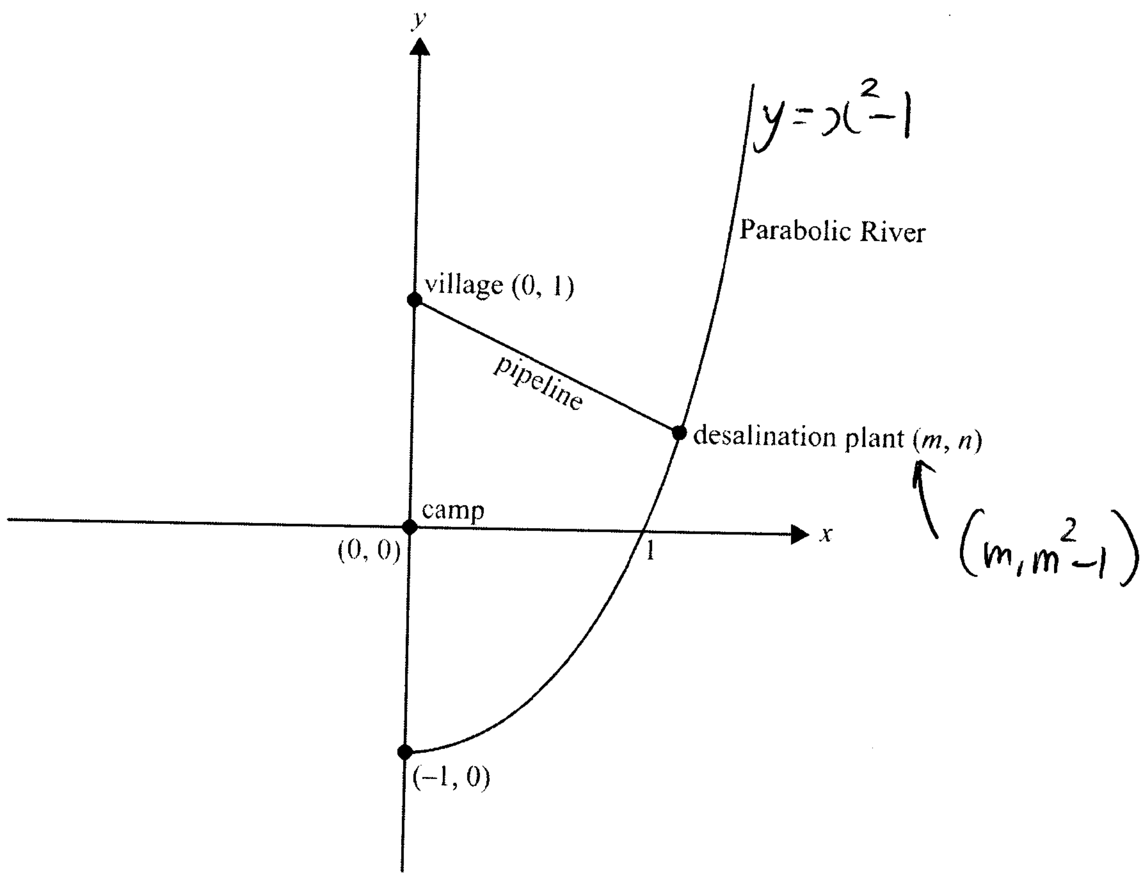
2 marks

Total 13 marks

Question 4

Deep in the South American jungle, Tasmania Jones has been working to help the Quetzacotl tribe to get drinking water from the very salty water of the Parabolic River. The river follows the curve with equation $y = x^2 - 1, x \geq 0$ as shown below. All lengths are measured in kilometres.

Tasmania has his camp site at (0, 0) and the Quetzacotl tribe's village is at (0, 1). Tasmania builds a desalination plant, which is connected to the village by a straight pipeline.



a. If the desalination plant is at the point (m, n) show that the length, L kilometres, of the straight pipeline that carries the water from the desalination plant to the village is given by

$$L = \sqrt{m^4 - 3m^2 + 4}.$$

$$L = \sqrt{(m-0)^2 + (m^2 - 1 - 1)^2}$$

$$= \sqrt{m^2 + (m^2 - 2)^2}$$

$$= \sqrt{m^2 + m^4 - 4m^2 + 4}$$

$$= \sqrt{m^4 - 3m^2 + 4}$$

3 marks

The desalination plant is actually built at $\left(\frac{\sqrt{7}}{2}, \frac{3}{4}\right)$.

If the desalination plant stops working, Tasmania needs to get to the plant in the minimum time.

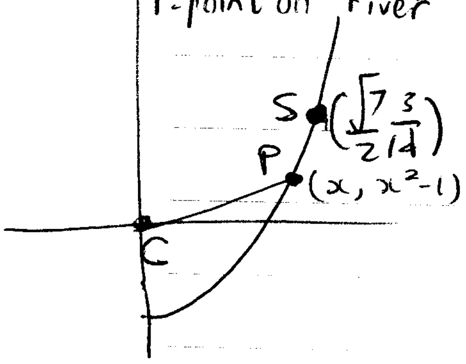
Tasmania runs in a straight line from his camp to a point (x, y) on the river bank where $x \le \frac{\sqrt{7}}{2}$. He then swims up the river to the desalination plant.

Tasmania runs from his camp to the river at 2 km per hour. The time that he takes to swim to the desalination plant is proportional to the difference between the y-coordinates of the desalination plant and the point where he enters the river.

c. Show that the total time taken to get to the desalination plant is given by

$T = \frac{1}{2}\sqrt{x^4 - x^2 + 1} + \frac{1}{4}k\left(7 - 4x^2\right)$ hours where k is a positive constant of proportionality.

S= deralination plant
P=point on river bank



$$T(x) = \frac{PC}{2} + k\left(\frac{3}{4} - (x^2-1)\right)$$

$$\therefore T(x) = \frac{\sqrt{(x-0)^2+(x^2-1)^2}}{2} + k\left(\frac{3}{4} - x^2 + 1\right)$$

$$T(x) = \frac{\sqrt{x^2+x^4-2x^2+1}}{2} + k\left(\frac{7}{4} - x^2\right)$$

$$T(x) = \frac{\sqrt{x^4-x^2+1}}{2} + k\left(\frac{7}{4} - x^2\right)$$

3 marks

The value of k varies from day to day depending on the weather conditions.

d. If $k = \frac{1}{2\sqrt{13}}$

i. find $\frac{dT}{dx}$

$$T(x) = \frac{\sqrt{x^4-x^2+1}}{2} + \frac{1}{2\sqrt{13}}\left(\frac{7}{4} - x^2\right)$$

$$\frac{dT}{dx} = \frac{x(2x^2-1)}{2\sqrt{x^4-x^2+1}} - \frac{\sqrt{13}\,x}{13}$$

b. If the desalination plant is built at the point on the river that is closest to the village

i. find $\frac{dL}{dm}$ and hence find the coordinates of the desalination plant

$$\frac{dL}{dm} = \frac{1}{2} \times \frac{1}{\sqrt{m^4 - 3m^2 + 4}} \times (4m^3 - 6m)$$

For a minimum, $\frac{dL}{dm} = 0$

$\therefore \quad 4m^3 - 6m = 0$

$\therefore \quad 2m(2m^2 - 3) = 0$

Since $m > 0$ $\quad m = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$

ii. find the length, in kilometres, of the pipeline from the desalination plant to the village.

$$L\left(\frac{\sqrt{6}}{2}\right) = \frac{\sqrt{7}}{2}$$

3 + 2 = 5 marks

ii. hence find the coordinates of the point where Tasmania should reach the river if he is to get to the desalination plant in the minimum time.

For a minimum, $\frac{dT}{dx} = 0$

$\therefore x = \frac{\sqrt{3}}{2}$ (since $x > 0$)

1 + 2 = 3 marks

e. On one particular day, the value of k is such that Tasmania should run directly from his camp to the point (1, 0) on the river to get to the desalination plant in the minimum time. Find the value of k on that particular day.

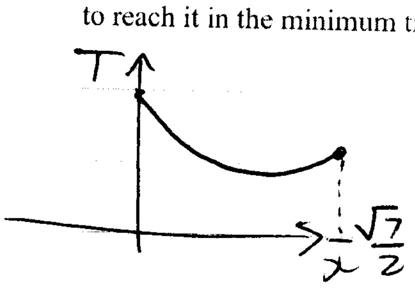
$T(x) = \frac{\sqrt{x^4 - x^2 + 1}}{2} + k\left(\frac{7}{4} - x^2\right)$

$T'(1) = 0$ (since there is a stationary point at $x = 1$)

Solving $T'(1) = 0$ for k gives: $k = \frac{1}{4}$

2 marks

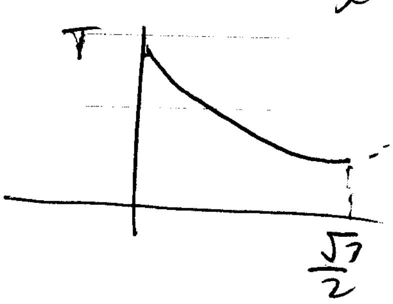
f. Find the values of k for which Tasmania should run directly from his camp towards the desalination plant to reach it in the minimum time.



$T(x) = \frac{\sqrt{x^4 - x^2 + 1}}{2} + k\left(\frac{7}{4} - x^2\right)$

Domain of $T(x)$ is $x \in \left[0, \frac{\sqrt{7}}{2}\right]$

If the minimum t/p of $T(x)$ occurs at $x \geq \frac{\sqrt{7}}{2}$, he should run directly to the plant.



$T'\left(\frac{\sqrt{7}}{2}\right) = \frac{5\sqrt{259}}{74} - k\sqrt{7}$

2 marks

Total 18 marks

If $T'\left(\frac{\sqrt{7}}{2}\right) \leq 0$ then the minimum turning point lies beyond $x = \frac{\sqrt{7}}{2}$ $\therefore$ $\frac{5\sqrt{259}}{74} - k\sqrt{7} \leq 0$

$\therefore k \geq \frac{5\sqrt{259}}{\sqrt{7} \times 74}$

$\therefore k \geq \frac{5\sqrt{37}}{74}$

Instructions

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working must be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let f: $[0, \infty) \rightarrow R, f(t) = 2e^{-t}$.

a. **i.** State the range of f.

$(0, \infty)$

ii. Find the rule for the inverse of f and state its domain.

$y = 2e^{-t}$

$t = 2e^{-y}$

$\frac{t}{2} = e^{-y}$

$-y = \log_e\left(\frac{t}{2}\right)$

$\therefore f^{-1}(t) = -\log_e\left(\frac{t}{2}\right),$

$t \in (0, \infty)$

dom(f) ran(f)

R $(0, \infty)$

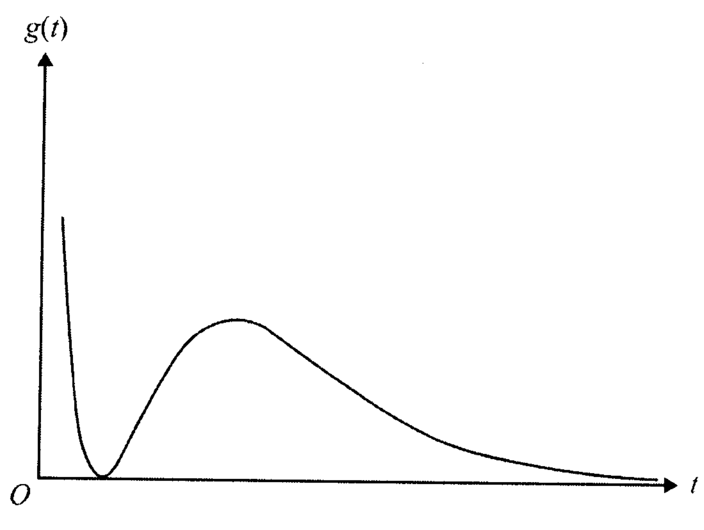
dom(f^{-1}) ran(f^{-1})

$(0, \infty)$ R

1 + 2 = 3 marks

b. Let g: $[0, \infty) \rightarrow R$, $g(t) = (t - 1)^2 e^{-t}$.

Part of the graph of g is shown.



Question 1 – continued

i. The rule for the derivative of g may be expressed in the form $g'(t) = (-t^2 + bt + c)e^{-t}$.
Find the exact values of b and c.

$g(t) = (t-1)^2 e^{-t}$

$\therefore g'(t) = -(t-3)(t-1)e^{-t}$

$= -e^{-t}(t^2 - 4t + 3)$

$= e^{-t}(-t^2 + 4t - 3)$

$\therefore b = 4, \quad c = -3$

ii. The graph of $y = g(t)$ has stationary points $(1, p)$ and (m, n).
Find the exact values of p, m and n.

For stationary points, $g'(t) = 0$

$\therefore -(t-3)(t-1)e^{-t} = 0$

$\therefore t = 1, 3$

If $t = 1$, $g(1) = 0 \quad \therefore (1, 0)$

If $t = 3$, $g(3) = (3-1)^2 \times e^{-3} = 4e^{-3}$

$\therefore (3, 4e^{-3})$

$\therefore p = 0, \quad m = 3, \quad n = \frac{4}{e^3}$

iii. For the function $q: [0, \infty) \to R$, with rule $q(t) = 2g(t) - 5$, state the exact coordinates of the stationary points of the graph of $y = q(t)$.

$q(t) = 2g(t) - 5$

For the transformation where $g(t)$ becomes

$q(t)$: $t' = t$

$y' = 2y - 5$

$\therefore (1, 0) \rightarrow (1, -5)$

$(3, \frac{4}{e^3}) \rightarrow (3, \frac{8}{e^3} - 5)$

3 + 2 + 2 = 7 marks

c. The function $h: R \rightarrow R$, $h(t) = (t^2 + at + 10)e^{-t}$, where a is a real constant, has derivative $h'(t) = (-t^2 + (2 - a)t + (a - 10))e^{-t}$.

Find the values of a such that

i. the graph of $y = h(t)$ has exactly one stationary point

For one stationary point, the quadratic equation: $-t^2 + (2-a)t + (a-10) = 0$ has only one solution $\therefore \Delta = 0$

$\therefore (2-a)^2 - 4(a-10) \times -1 = 0$

$(2-a)^2 + 4(a-10) = 0$

$4 - 4a + a^2 + 4a - 40 = 0$

$\therefore a^2 - 36 = 0$

$a = \pm 6.$

ii. $h'(t) < 0$ for all $t \in R$.

$h'(t) < 0$ for all $t \in \mathbb{R}$ if

$-t^2 + (2-a)t + (a-10) = 0$ has no solution

$\therefore$

$\Delta < 0$

$\therefore a^2 - 36 < 0$

$\therefore -6 < a < 6.$

3 + 2 = 5 marks

Total 15 marks

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Two ships, the Elsa and the Violet, have collided. Fuel immediately starts leaking from the Elsa into the sea. The captain of the Elsa estimates that at the time of the collision his ship has 6075 litres of fuel on board and he also forecasts that it will leak into the sea at a rate of $\frac{t^2}{5}$ litres per minute, where t is the number of minutes that have elapsed since the collision.

a. At this rate how long, in minutes, will it take for all the fuel from the Elsa to leak into the sea?

Let V = volume of oil in the sea

Variables: V, t

$$\frac{dV}{dt} = \frac{t^2}{5}$$

$$\therefore V = \int \frac{t^2}{5}\,dt$$

$$V = \frac{t^3}{15} + c$$

When $t = 0$, $V = 0$ $\therefore c = 0$

$$\therefore V = \frac{t^3}{15}$$

3 marks

When $V = 6075$

$$6075 = \frac{t^3}{15}$$

$$t = \sqrt[3]{15 \times 6075}$$

$t = 45$ $\therefore$ 45 minutes

In fact, the captain's assessment of the situation is wrong. The amount of fuel on board at the time of the collision **is not** 6075 litres. The fuel actually leaks into the sea forming a circular oil slick. The area of this circle is increasing at the constant rate of 20 square metres per minute.

At time t minutes after the collision the radius of the circle is r metres.

b. Show that when the radius of the circle is 3 metres, the radius is increasing at $\frac{10}{3\pi}$ metres per minute.

Variables, r, t, A

$$\frac{dr}{dt} = \frac{dr}{dA} \cdot \frac{dA}{dt}$$

$$\frac{dA}{dt} = 20$$

$$A = \pi r^2 \quad \therefore \frac{dA}{dr} = 2\pi r$$

$$\therefore \frac{dr}{dA} = \frac{1}{2\pi r}$$

$$\therefore \frac{dr}{dt} = \frac{1}{2\pi r} \times 20$$

$$= \frac{10}{\pi r}$$

When $r = 3$,

$$\frac{dr}{dt} = \frac{10}{3\pi} \quad m/min$$

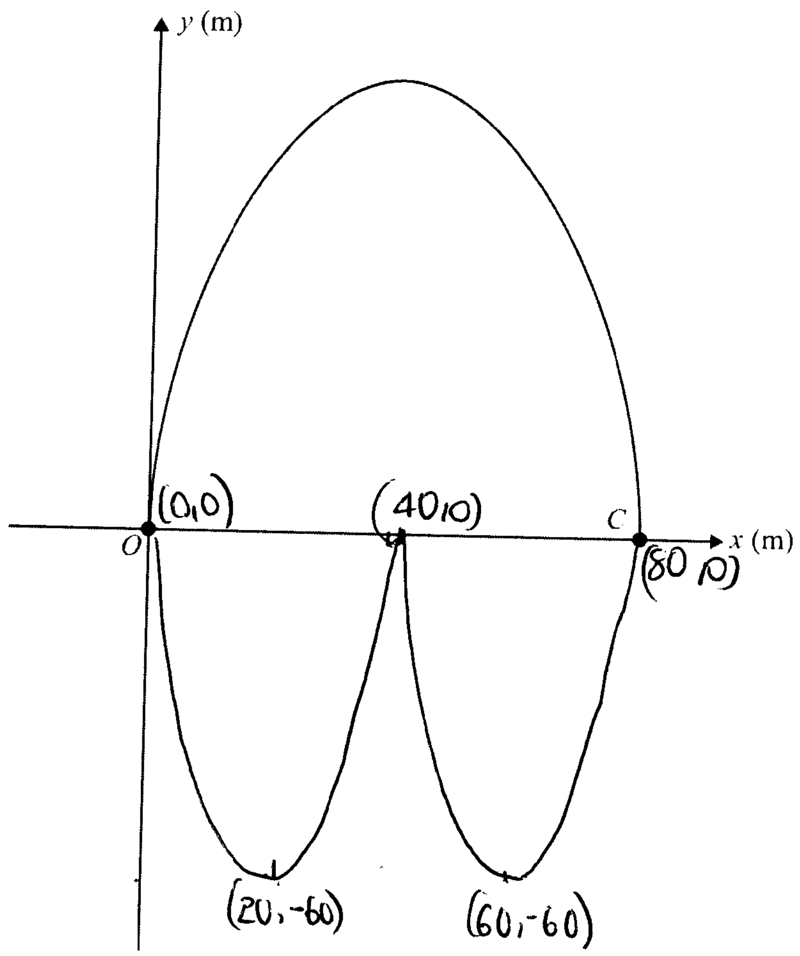
3 marks

In the following, all measurements of distance are in metres.

Sometime later, the fuel stops leaking. Authorities will use two floating barriers to surround the oil slick. One of the barriers, barrier 1, is in the shape of the graph of the function

$$g: [0, 80] \to R,\ g(x) = 75\sin\left(\frac{\pi x}{80}\right).$$

This graph is shown below.



The second barrier enclosing the oil slick, barrier 2, is in the shape of the graph of the function h given by

$$h: [0, 80] \to R,\ h(x) = -60\left|\sin\left(\frac{\pi x}{40}\right)\right|.$$

c. On the axes above, sketch the graph of $y = h(x)$. Label all axes intercepts and turning points with their coordinates.

2 marks

d. Some time later, the oil slick completely fills the area enclosed by the two barriers.

i. Find $\int_0^{40} \sin\left(\frac{\pi x}{40}\right) dx$

$$\int_0^{40} \sin\left(\frac{\pi x}{40}\right) dx = \left[\frac{-40}{\pi}\cos\left(\frac{\pi x}{40}\right)\right]_0^{40}$$

$$= \frac{40}{\pi} + \frac{40}{\pi} = \frac{80}{\pi}$$

ii. Find, to the nearest square metre, the total area of the oil slick at that time.

For $0 \le x \le 40$, $-60\left|\sin\left(\frac{\pi x}{40}\right)\right|$

$$= -60\sin\left(\frac{\pi x}{40}\right)$$

Required area = by symmetry:

$$2\int_0^{40} 75\sin\left(\frac{\pi x}{80}\right) - -60\sin\left(\frac{\pi x}{40}\right) dx$$

$$= 2\int_0^{40} 75\sin\left(\frac{\pi x}{80}\right) + 60\sin\left(\frac{\pi x}{40}\right) dx$$

$$= \frac{21600}{\pi}$$

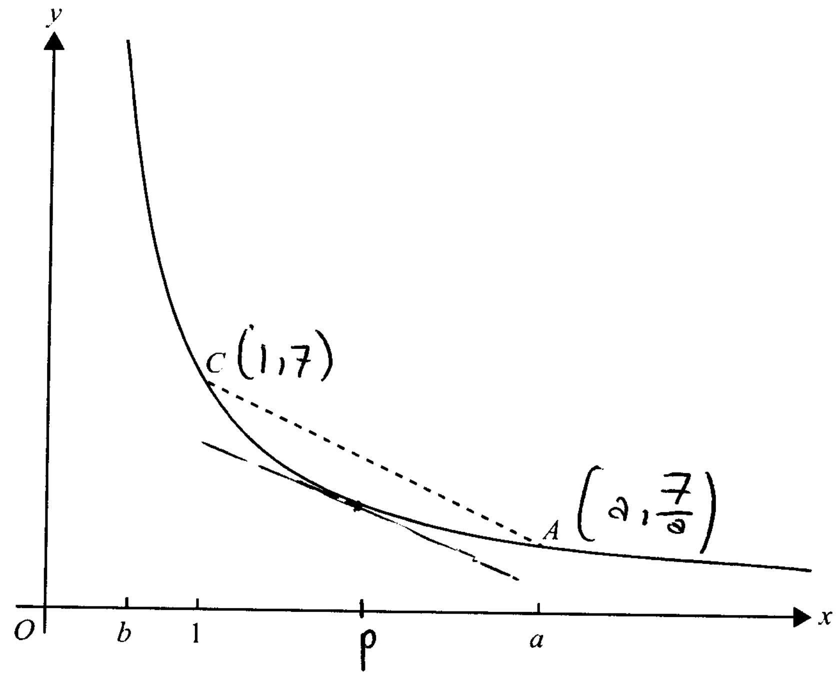
$$\approx 6875 \text{ m}^2$$

1 + 2 = 3 marks

Total 11 marks

Question 2

The diagram below shows part of the graph of the function $f: R^+ \rightarrow R,\ f(x) = \frac{7}{x}$.



The line segment CA is drawn from the point $C(1, f(1))$ to the point $A(a, f(a))$ where $a > 1$.

a. **i.** Calculate the gradient of CA in terms of a.

$$m = \frac{\frac{7}{a} - 7}{a - 1} = \frac{7\left(\frac{1}{a} - 1\right)}{a - 1} = \frac{7\frac{(1-a)}{a}}{(a-1)}$$

$$= -\frac{7}{a}$$

ii. At what value of x between 1 and a does the tangent to the graph of f have the same gradient as CA?

$$f(x) = \frac{7}{x}$$

$$\therefore f'(x) = \frac{-7}{x^2}$$

1 + 2 = 3 marks

At required point where $x = p$

$$f'(p) = \frac{-7}{p^2}$$

$$= \frac{-7}{a}$$

$$\therefore \frac{-7}{a} = \frac{-7}{p^2} \qquad \therefore p = \sqrt{a}$$

b. **i.** Calculate $\int_1^e f(x)dx$.

$$\int_1^e f(x)\,dx = \int_1^e \frac{7}{x}\,dx$$

$$= \left[7\log_e x\right]_1^e = 7\log_e e - 7\log_e 1 = 7$$

ii. Let b be a positive real number less than one. Find the exact value of b such that $\int_b^1 f(x)dx$ is equal to 7.

$$\int_b^1 f(x)\,dx = 7$$

$$\therefore \left[7\log_e x\right]_b^1 = 7 \quad \therefore 7\log_e 1 - 7\log_e b = 7$$

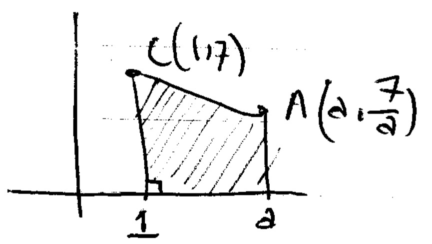
$$-7\log_e b = 7$$

$$\therefore \log_e b = -1$$

$$\therefore b = \frac{1}{e}$$

1 + 2 = 3 marks

c. **i.** Express the area of the region bounded by the line segment CA, the x-axis, the line $x = 1$ and the line $x = a$ in terms of a.



Required area = A trapezium

$$= \frac{1}{2}(a-1)\left(\frac{7}{a}+7\right)$$

$$= \frac{7}{2}(a-1)\left(\frac{1}{a}+1\right)$$

ii. For what exact value of a does this area equal 7?

$$\frac{7}{2}(a-1)\left(\frac{1}{a}+1\right) = 7$$

$$\therefore (a-1)\left(\frac{1}{a}+1\right) = 2$$

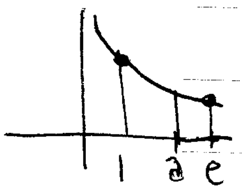
$$\therefore a = \sqrt{2}+1, \text{ since } a > 1$$

iii. Using the value for a determined in **c.ii.**, explain in words, without evaluating the integral, why $\int_1^a f(x)dx < 7$.

Use this result to explain why $a < e$.

Because the curve $y = f(x)$ lies below the segment CA for all $x \in (1, a)$

Since $\int_1^e f(x)dx = 7$ and since $\int_1^a f(x)dx < 7$



$\Rightarrow \quad a < e$

$\therefore \sqrt{2}+1 < e$

2 + 2 + 1 = 5 marks

d. Find the exact values of m and n such that $\int_1^{mn} f(x)dx = 3$ and $\int_1^{\frac{m}{n}} f(x)dx = 2$.

$$\int_1^{mn} f(x)dx = \left[7\log_e x\right]_1^{mn}$$
$$= 7\log_e(mn) - 7\log_e 1$$
$$= 7\log_e(mn)$$

$$\int^{\frac{m}{n}} f(x)dx = 7\log_e\left(\frac{m}{n}\right)$$

2 marks

Total 13 marks

$\therefore 7\log_e(mn) = 3 \quad (1)$

$7\log_e\left(\frac{m}{n}\right) = 2 \quad (2)$

Solving:

$m = -e^{\frac{5}{14}}, n = -e^{\frac{1}{14}}$

or

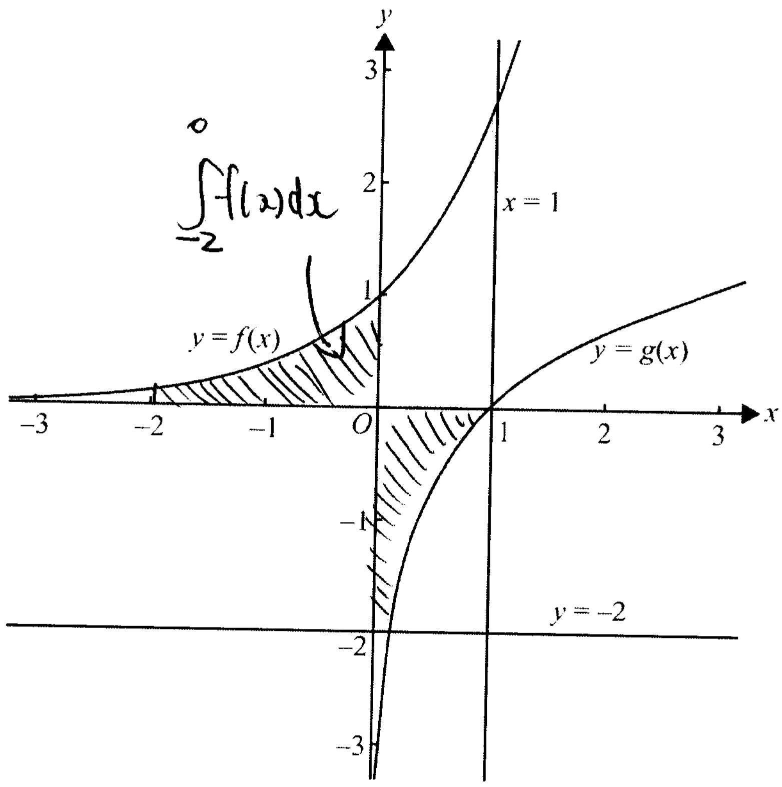
$m = e^{\frac{5}{14}}, n = e^{\frac{1}{14}}$

Question 5

The shaded region in the diagram below is the plan of a mine site for the Black Possum mining company. All distances are in kilometres.

Two of the boundaries of the mine site are in the shape of the graphs of the functions

$f: R \to R, f(x) = e^x$ and $g: R^+ \to R, g(x) = \log_e(x)$.



a. **i.** Evaluate $\int_{-2}^{0} f(x)dx$.

$$\int_{-2}^{0} e^x \, dx = \left[e^x\right]_{-2}^{0}$$
$$= e^0 - e^{-2}$$
$$= 1 - \frac{1}{e^2}$$

ii. **Hence**, or otherwise, find the area of the region bounded by the graph of g, the x and y axes, and the line $y = -2$.

Since $g(x) = f^{-1}(x)$ the required area is equal to $\int_{-2}^{0} e^x \, dx = 1 - \frac{1}{e^2}$

iii. Find the **total** area of the shaded region.

$$\text{Total} = \int_0^1 e^x\,dx + 1 - \frac{1}{e^2}$$
$$= \left[e^x\right]_0^1 + 1 - \frac{1}{e^2}$$
$$e - 1 + 1 - \frac{1}{e^2}$$
$$= e - \frac{1}{e^2}$$

1 + 1 + 1 = 3 marks

b. The mining engineer, Victoria, decides that a better site for the mine is the region bounded by the graph of g and that of a new function $k: (-\infty, a) \to R$, $k(x) = -\log_e(a - x)$, where a is a positive real number.

i. Find, in terms of a, the x-coordinates of the points of intersection of the graphs of g and k.

$$g(x) = \log_e(x)$$
$$k(x) = -\log_e(a - x) = \log_e\left(\frac{1}{a - x}\right)$$

At the intersection, $x = \frac{1}{a - x}$

$$\therefore ax - x^2 = 1$$
$$x^2 - ax + 1 = 0$$
$$x = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

ii. **Hence**, find the set of values of a, for which the graphs of g and k have two distinct points of intersection.

For two distinct intersection points,

$$a^2 - 4 > 0$$
$$\therefore \{a : a > 2\}$$

2 + 1 = 3 marks

c. For the new mine site, the graphs of g and k intersect at two distinct points, A and B. It is proposed to start mining operations along the line segment AB, which joins the two points of intersection.

Victoria decides that the graph of k will be such that the x-coordinate of the midpoint of AB is $\sqrt{2}$. Find the value of a in this case.

Find A:

When $x = \dfrac{a-\sqrt{a^2-4}}{2}$, $y = \log_e\left(\dfrac{a-\sqrt{a^2-4}}{2}\right)$

$$\therefore A = \left(\frac{a-\sqrt{a^2-4}}{2}, \log_e\left(\frac{a-\sqrt{a^2-4}}{2}\right)\right)$$

Likewise, $B = \left(\dfrac{a+\sqrt{a^2-4}}{2}, \log_e\left(\dfrac{a+\sqrt{a^2-4}}{2}\right)\right)$

x co-ord. of Midpoint of AB

$$= \frac{1}{2}\left(\frac{a-\sqrt{a^2-4}}{2} + \frac{a+\sqrt{a^2-4}}{2}\right)$$

2 marks

$$= \frac{a}{2}$$

$$\therefore \frac{a}{2} = \sqrt{2}$$

$$\therefore a = 2\sqrt{2}$$

END OF QUESTION AND ANSWER BOOK

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

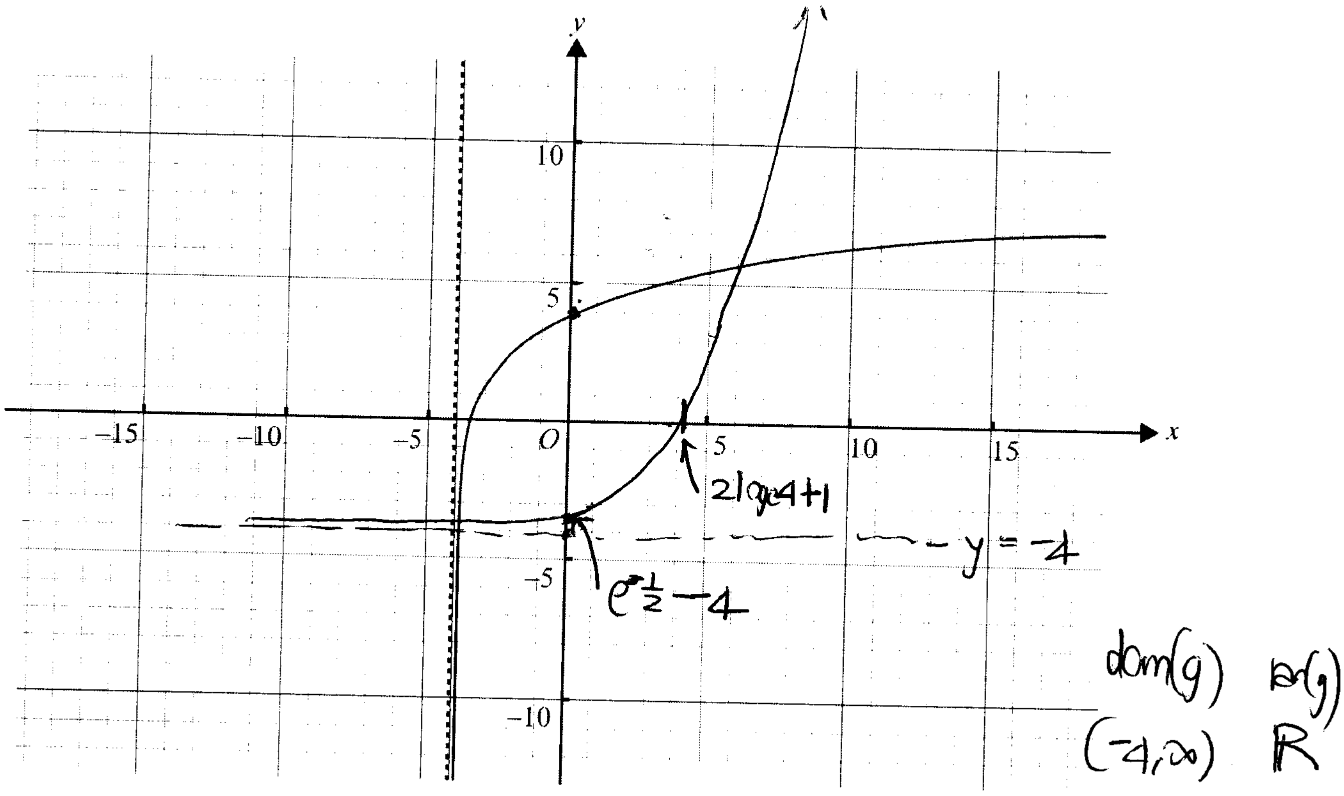
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

a. Part of the graph of the function $g: (-4, \infty) \to R$, $g(x) = 2\log_e(x+4) + 1$ is shown on the axes below.



i. Find the rule and domain of g^{-1}, the inverse function of g.

$y = 2\log_e(x+4) + 1$

$x = 2\log_e(y+4) + 1$

$\therefore \frac{x-1}{2} = \log_e(y+4)$

$y + 4 = e^{\frac{x-1}{2}}$

$\therefore g^{-1}(x) = e^{\frac{x-1}{2}} - 4$, $x \in R$

ii. On the set of axes above sketch the graph of g^{-1}. Label the axes intercepts with their exact values.

x int: $(2\log_e 4 + 1, 0)$

y-int: $(0, e^{-\frac{1}{2}} - 4)$

iii. Find the values of x, correct to three decimal places, for which $g^{-1}(x) = g(x)$.

$y = x$ (1)

$y = 2\log_e(x+4)+1$ (2)

$x = -3.914, \ 5.503$

iv. Calculate the area enclosed by the graphs of g and g^{-1}. Give your answer correct to two decimal places.

$$\int_{-3.914}^{5.503} 2\log_e(x+4)+1 - \left(e^{\frac{x-1}{2}}-4\right) dx$$

$$= \int_{-3.914}^{5.503} 2\log_e(x+4) - e^{\frac{x-1}{2}} + 5 \; dx$$

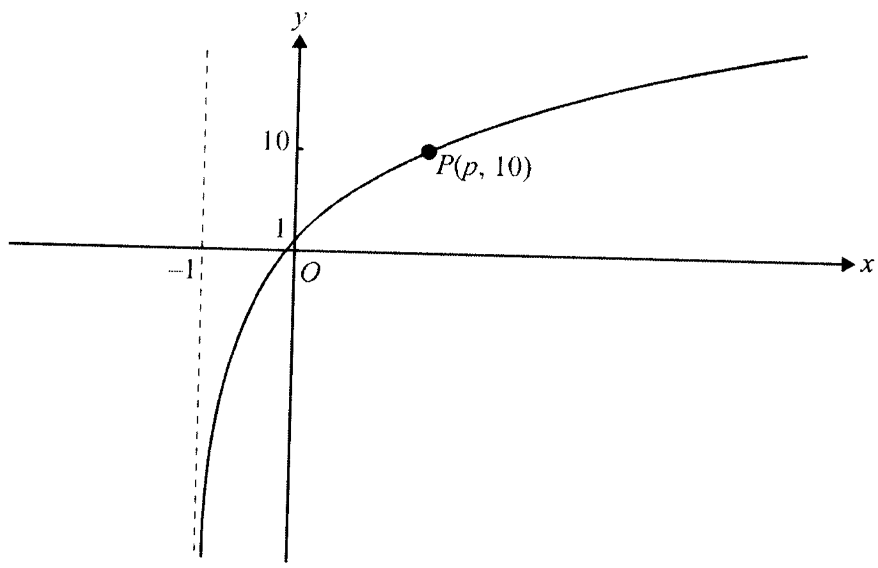
$= 52.63$ sq. units

3 + 3 + 2 + 2 = 10 marks

b. The diagram below shows part of the graph of the function with rule

$$f(x) = k\log_e(x + a) + c, \text{ where } k, a \text{ and } c \text{ are real constants.}$$

- The graph has a vertical asymptote with equation $x = -1$.
- The graph has a y-axis intercept at 1.
- The point P on the graph has coordinates $(p, 10)$, where p is another real constant.



i. State the value of a.

$a = 1$

ii. Find the value of c.

$y = k\log_e(x+1) + c$

When $x = 0$, $y = 1$ $\therefore$ $1 = k\log_e 1 + c$

$\therefore c = 1$

iii. Show that $k = \dfrac{9}{\log_e(p+1)}$.

$\therefore y = k\log_e(x+1) + 1$

When $x = p$, $y = 10$

$\therefore 10 = k\log_e(p+1) + 1$

$\therefore 9 = k\log_e(p+1)$

$k = \dfrac{9}{\log_e(p+1)}$

iv. Show that the gradient of the tangent to the graph of f at the point P is $\dfrac{9}{(p+1)\log_e(p+1)}$.

$$f(x) = \frac{9}{\log_e(p+1)} \log_e(x+1) + 1$$

$$\therefore f'(x) = \frac{9}{\log_e(p+1)} \times \frac{1}{x+1}$$

When $x = p$, $\quad f'(p) = \dfrac{9}{(p+1)(\log_e(p+1))}$

v. If the point $(-1, 0)$ lies on the tangent referred to in **part b.iv.**, find the exact value of p.

If $(-1,0)$ lies on the tangent, then gradient of tangent can be fand from $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

since $(x_1, y_1) = (p, 10)$ and $(x_2, y_2) = (-1, 0)$

$$\therefore m = \frac{0-10}{-1-p} = \frac{-10}{-1-p} = \frac{10}{1+p}$$

1 + 1 + 2 + 1 + 2 = 7 marks

Total 17 marks

But $m = f'(p)$

$$= \frac{9}{(p+1)\log_e(p+1)}$$

$$\therefore \frac{10}{(p+1)} = \frac{9}{(p+1)\log_e(p+1)}$$

$$\therefore 10 = \frac{9}{\log_e(p+1)}$$

$$\therefore \log_e(p+1) = \frac{9}{10}$$

$$p+1 = e^{9/10}$$

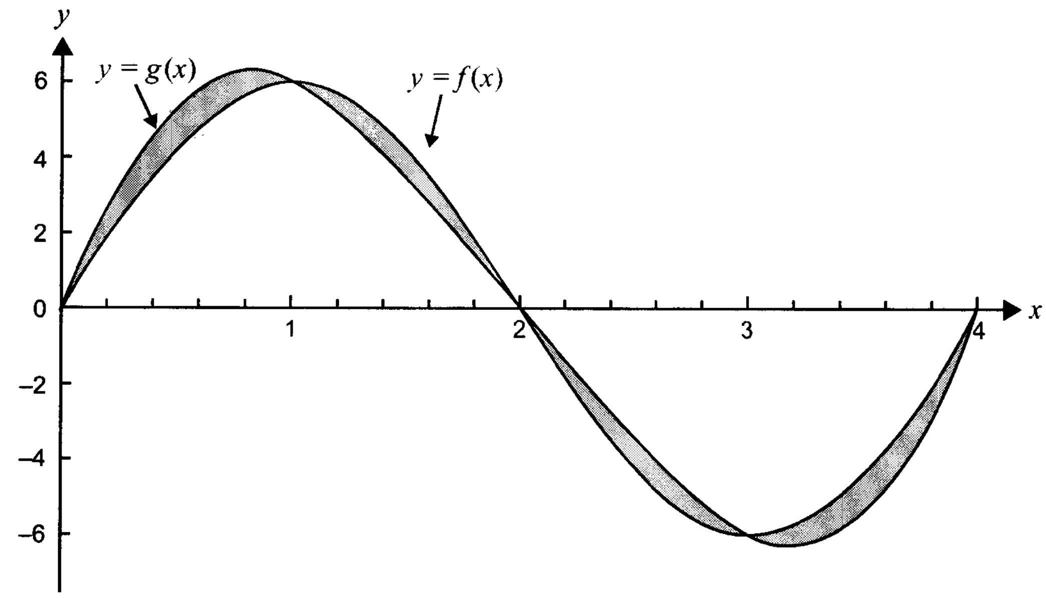
$$\therefore p = e^{9/10} - 1$$

Question 3

Shown below are the graphs of the functions

$$f: [0, 4] \to R, f(x) = 6\sin\left(\frac{\pi x}{2}\right) \text{ and}$$
$$g: [0, 4] \to R, g(x) = 2x(x-2)(x-4) = 2(x^3 - 6x^2 + 8x)$$

The point (1, 6) lies on both graphs.



a. Find the exact value of x for which $g(x)$ is maximum.

$g(x) = 2x(x-2)(x-4)$

$g'(x) = 6x^2 - 24x + 16$

Let $g'(x) = 0$ for a stationary point

$\therefore 6x^2 - 24x + 16 = 0$

$x = \dfrac{-2(\sqrt{3}-3)}{3}, \dfrac{2(\sqrt{3}+3)}{3}$

But the maximum t/p lies in the interval

2 marks

$x \in [0, 2] \quad \therefore \quad x = \dfrac{-2(\sqrt{3}-3)}{3}$

$= \dfrac{2(3-\sqrt{3})}{3}$

b. **i.** Write an expression for the total area of the shaded regions using definite integrals.

$$\int_0^1 2(x^3-6x^2+8x)-6\sin\left(\frac{\pi x}{2}\right)dx + \int_1^2 6\sin\left(\frac{\pi x}{2}\right)-2(x^3-6x^2+8x)\,dx$$
$$+\int_2^3 2(x^3-6x^2+8x)-6\sin\left(\frac{\pi x}{2}\right)dx + \int_3^4 6\sin\left(\frac{\pi x}{2}\right)-2(x^3-6x^2+8x)\,dx$$

ii. Find the total area of these shaded regions.

Let $V(x) = 2(x^3-6x^2+8x)-6\sin\left(\frac{\pi x}{2}\right)$

Required area $= \int_0^1 V(x)\,dx + \int_1^2 -V(x)\,dx + \int_2^3 V(x)\,dx + \int_3^4 -V(x)\,dx$

$= 2\left(\frac{9}{2}-\frac{12}{\pi}\right)+2\left(\frac{12}{\pi}-\frac{7}{2}\right) = 2$ sq. units

2 + 2 = 4 marks

c. Find, correct to two decimal places, the maximum value of $|f(x)-g(x)|$ for $0 \le x \le 4$ and the values of x in this interval for which this maximum occurs.

$V(x) = 2(x^3-6x^2+8x) - 6\sin\left(\frac{\pi x}{2}\right)$

Let $V'(x)=0$ for a stationary point.

Maximum value of $|f(x)-g(x)| = |h(x)|$

is 1.08 at $x = 0.38, 3.62$

2 marks

d. Let $h: [-3, 1] \to R, h(x) = 6\cos\left(\frac{\pi x}{2}\right)$.

i. State a sequence of two transformations which takes the graph of f to the graph of h.

Reflect in x-axis

Translate 3 units in negative x direction.

[This can be seen by graphing $f(x) = 6\sin\left(\frac{\pi x}{2}\right)$ and $h(x) = 6\cos\left(\frac{\pi x}{2}\right)$ on CAS

OR

$-6\sin\left(\frac{\pi}{2}(x+3)\right) = -6\sin\left(\frac{\pi x}{2}+\frac{3\pi}{2}\right)$

$= 6\cos\left(\frac{\pi x}{2}\right)$ because $\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$]

ii. Hence or otherwise, find a cubic polynomial function with domain $[-3, 1]$ that has the same x-intercepts as h and the same maximum and minimum values as g.

So, we take the function $g(x) = 2(x^3 - 6x^2 + 8x)$ and carry out the transformations:

reflect in x-axis $\quad -g(x)$

then translate 3 units to left: $\quad -g(x+3)$.

2 + 2 = 4 marks

Total 12 marks

$$g(x) = 2x(x-2)(x-4)$$

$$\therefore -g(x+3) = -2(x+3)(x+3-2)(x+3-4)$$

$$= -2(x+3)(x+1)(x-1).$$

Required cubic polynomial is:

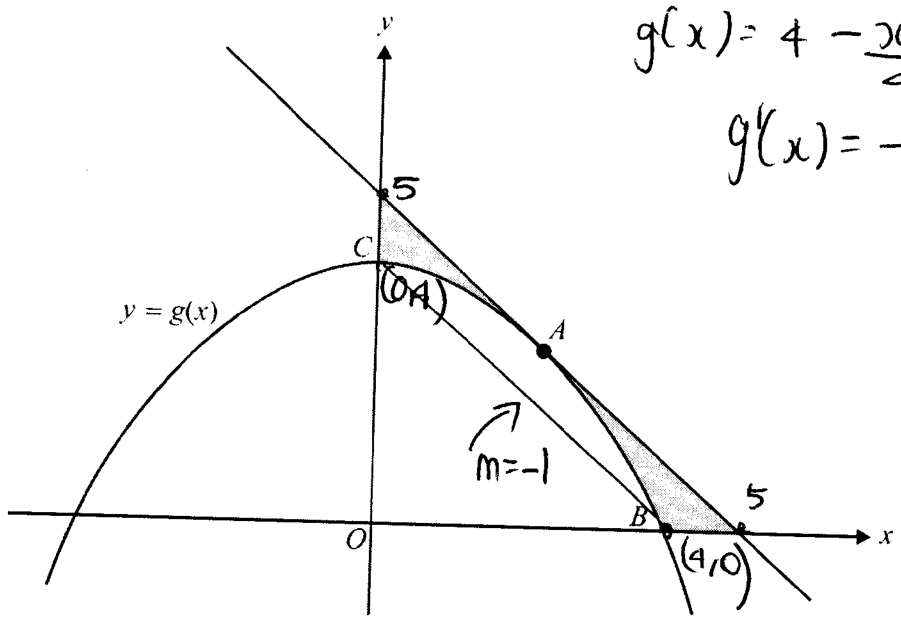
$$P(x) = -2(x+3)(x+1)(x-1)$$

Question 4 (16 marks)

Part of the graph of a function $g: R \to R$, $g(x) = \dfrac{16 - x^2}{4}$ is shown below.

$g(x) = 4 - \dfrac{x^2}{4}$

$g'(x) = -\dfrac{x}{2}$



a. Points B and C are the positive x-intercept and y-intercept of the graph of g, respectively, as shown in the diagram above. The tangent to the graph of g at the point A is parallel to the line segment BC.

i. Find the equation of the tangent to the graph of g at the point A. 2 marks

$g'(x) = -\dfrac{x}{2}$ $\quad \therefore -\dfrac{x}{2} = -1$ at A

$\therefore x = 2$

$g(2) = 4 - \dfrac{2^2}{4} = 3$ $\quad \therefore A = (2,3)$

Equation of tangent: $\quad y - 3 = -(x-2)$

$y = -x + 5$

ii. The shaded region shown in the diagram above is bounded by the graph of g, the tangent at the point A, and the x-axis and y-axis.

Evaluate the area of this shaded region. 3 marks

Shaded area = A triangle $- \displaystyle\int_0^4 \left(4 - \frac{x^2}{4}\right) dx$

$= \dfrac{1}{2} \times 5^2 - \left[4x - \dfrac{x^3}{12}\right]_0^4$

$= \dfrac{25}{2} - \left[\left(16 - \dfrac{64}{12}\right) - 0\right]$

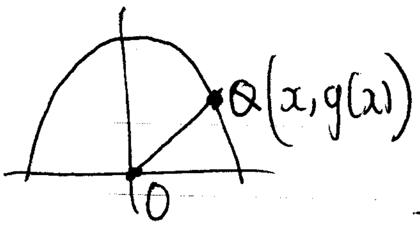
$= \dfrac{25}{2} - \left(\dfrac{48}{3} - \dfrac{16}{3}\right)$

$= \dfrac{25}{2} - \dfrac{32}{3} = \dfrac{75}{6} - \dfrac{64}{6} = \dfrac{11}{6}$ sq. units

b. Let Q be a point on the graph of $y = g(x)$.

Find the positive value of the x-coordinate of Q, for which the distance OQ is a minimum and find the minimum distance.

3 marks



$d(OQ) = \sqrt{x^2 + [g(x)]^2}$

Let $f(x) = d(OQ)$

$f(x) = \sqrt{x^2 + \left(4 - \frac{x^2}{4}\right)^2}$

$f(x) = \sqrt{x^2 + 16 - 2x^2 + \frac{x^4}{16}}$

$\therefore f(x) = \sqrt{\frac{x^4}{16} - x^2 + 16}$ For a stationary point, $f'(x) = 0$

$\therefore \dfrac{\frac{x^3}{4} - 2x}{2\sqrt{\frac{x^4}{16} - x^2 + 16}} = 0$ $\therefore \frac{x^3}{4} - 2x = 0$

$x\left(\frac{x^2}{4} - 2\right) = 0$

$x = 0$ or $x = \sqrt{8}$

since $x > 0, x = \sqrt{8}$

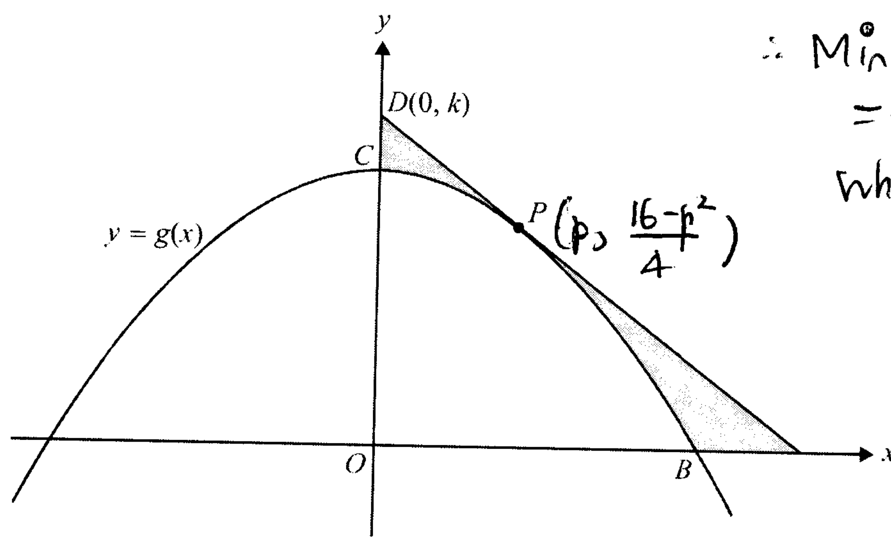
$f(\sqrt{8}) = \sqrt{8 + (4-2)^2}$

$= \sqrt{12}$

$= 2\sqrt{3}$

$\therefore$ Min distance $= 2\sqrt{3}$ when $x = 2\sqrt{2}$

The tangent to the graph of g at a point P has a **negative** gradient and intersects the y-axis at point $D(0, k)$, where $5 \le k \le 8$.



$P\left(p, \frac{16-p^2}{4}\right)$

c. Find the gradient of the tangent in terms of k.

2 marks

$f'(p) = \dfrac{\frac{16-p^2}{4} - k}{p - 0}$

$\therefore \dfrac{-p}{2} = \dfrac{\frac{16-p^2}{4} - k}{p}$

$\dfrac{-p^2}{2} = 4 - \dfrac{p^2}{4} - k$

$\therefore k = 4 + \dfrac{p^2}{4}$

$\dfrac{p^2}{4} = k - 4$

$\therefore p^2 = 4(k-4)$

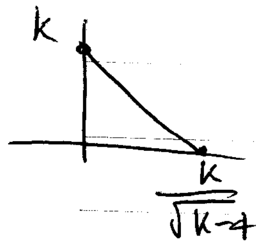
$\therefore p = 2\sqrt{k-4}$, $p > 0$

$\therefore f'(p) = \dfrac{-p}{2} = -\sqrt{k-4}$

Gradient $= -\sqrt{k-4}$

d. **i.** Find the rule $A(k)$ for the function of k that gives the area of the shaded region. 2 marks

$A(k) = \text{Atnangle} - \int_0^4 \left(4 - \frac{x^2}{4}\right) dx$



Eqn of tangent: $y = -\sqrt{k-4}\, x + k$

x-int: let $y=0$ $\therefore$ $x = \frac{k}{\sqrt{k-4}}$

$\therefore A(k) = \frac{1}{2} k \cdot \frac{k}{\sqrt{k-4}} - \frac{32}{3} = \frac{k^2}{2\sqrt{k-4}} - \frac{32}{3}$

$\therefore A(k) = \frac{k^2}{2\sqrt{k-4}} - \frac{32}{3}$

ii. Find the **maximum** area of the shaded region and the value of k for which this occurs. 2 marks

Maximum area $= A(8)$

(occurs at endpoint)

$= \frac{64}{2 \times \sqrt{4}} - \frac{32}{3}$

$= \frac{64}{4} - \frac{32}{3}$

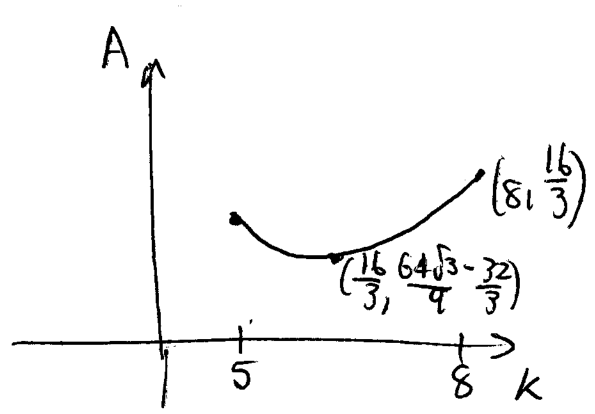
$= \frac{16}{3}$

iii. Find the **minimum** area of the shaded region and the value of k for which this occurs. 2 marks

$\frac{dA}{dk} = 0$ for a stationary point

$\therefore k = 0, \frac{16}{3}$

But $5 \leq k \leq 8$ $\therefore$ Stationary point occurs at $k = \frac{16}{3}$



$A\left(\frac{16}{3}\right) = \frac{64\sqrt{3}}{9} - \frac{32}{3}$

END OF QUESTION AND ANSWER BOOK