ANSWERS

EXERCISE SHEET for May 1

Question 1

For the function $g(x) = 2^{3x}$, show that $g(x + y) = g(x)g(y)$

$$g(x+y) = 2^{3(x+y)}$$
$$= 2^{3x+3y}$$
$$= 2^{3x} \cdot 2^{3y}$$

$$g(x) \cdot g(y)$$
$$= 2^{3x} \cdot 2^{3y}$$
$$\therefore g(x+y) = g(x) \cdot g(y)$$

Question 2

For the function $f(x) = log_{10}(x)$ if $x > 0$ show that:

$f(\sqrt{x}) = \frac{1}{2}f(x)$

$$f(\sqrt{x}) = \log_{10}(\sqrt{x})$$
$$= \log_{10}(x^{1/2})$$
$$= \frac{1}{2}\log_{10} x$$
$$\frac{1}{2}f(x) = \frac{1}{2}\log_{10} x \quad \therefore \frac{1}{2}f(x) = f(\sqrt{x})$$

Question 3

Let $f: R \to R$, $f(x) = x^2$.

Which one of the following is not true?

A. $f(xy) = f(x)f(y)$

B. $f(x) - f(-x) = 0$

C. $f(2x) = 4f(x)$

D. $f(x - y) = f(x) - f(y)$

E. $f(x + y) + f(x - y) = 2(f(x) + f(y))$

$$f(x-y) = (x-y)^2$$
$$= x^2 - 2xy + y^2$$
$$f(x) - f(y) = x^2 - y^2$$
$$\therefore f(x-y) \neq f(x) - f(y)$$

Question 4

Let $f: R \to R$, $f(x) = e^x + e^{-x}$.
For all $u \in R$, $f(2u)$ is equal to

A. $f(u) + f(-u)$

B. $2f(u)$

C. $(f(u))^2 - 2$

D. $(f(u))^2$

E. $(f(u))^2 + 2$

$f(2u) = e^{2u} + e^{-2u}$

$(f(u))^2 - 2 = (e^u + e^{-u})^2 - 2$

$= (e^u)^2 + 2e^u \cdot e^{-u} + e^{-2u} - 2$

$= e^{2u} + 2 + e^{-2u} - 2$

$= e^{2u} + e^{-2u}$

$= f(2u)$

Question 5

A function f has the following two properties for all real values of θ.

$$f(\pi - \theta) = -f(\theta) \text{ and } f(\pi - \theta) = -f(-\theta)$$

A possible rule for f is

A. $f(x) = \sin(x)$

B. $f(x) = \cos(x)$

C. $f(x) = \tan(x)$

D. $f(x) = \sin\left(\frac{x}{2}\right)$

E. $f(x) = \tan(2x)$

Question 6

The function $h(x) = \sin(x)$ is transformed according to the matrix equation:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$x' = -3x$ $\quad \therefore x = \frac{x'}{-3}$

$y' = \frac{1}{4}y$ $\quad y = 4y'$

a. Write down the equation of the transformed function.

$y = \sin(x)$

$4y' = \sin\left(-\frac{x'}{3}\right)$

New equation: $y = \frac{1}{4}\sin\left(-\frac{x}{3}\right)$

Q 6 (b) Period $= \frac{2\pi}{\frac{1}{3}} = 6\pi$

Amp $= \frac{1}{4}$

Q7. (a)

$x' = \frac{1}{3}x + 1 \quad \therefore x = 3(x' - 1)$

$y' = -2y - 2 \quad \therefore y = \frac{y' + 2}{-2}$

New rule:

$\frac{y' + 2}{-2} = e^{3(x'-1)}$

$\therefore y' + 2 = -2e^{3(x'-1)}$

$\therefore y' = -2e^{3(x'-1)} - 2$

$\therefore y = -2e^{3(x-1)} - 2$

$\therefore f(x) = -2e^{3(x-1)} - 2$

(b) Dilate by factor $\frac{1}{3}$ from y-axis

Dilate by factor 2 from x-axis

Reflect in x-axis

Translate 1 unit to right and 2 units down

(c) Maximal domain $= \mathbb{R}$

Range $= (-\infty, -2)$

(d)

dom(f)	ran(f)
$\mathbb{R}$	$(-\infty, -2)$
dom(f^{-1})	ran(f^{-1})
$(-\infty, -2)$	$\mathbb{R}$

$$y = -2e^{3(x-1)} - 2$$

$$\downarrow$$

$$x = -2e^{3(y-1)} - 2$$

$$\frac{x+2}{-2} = e^{3y-3}$$

$$\therefore \log_e\left(\frac{x+2}{-2}\right) = 3y-3$$

$$y = \frac{1}{3}\log_e\left(\frac{x+2}{-2}\right) + 1$$

$$\therefore \quad f^{-1}(x) = \frac{1}{3}\log_e\left(\frac{x+2}{-2}\right) + 1, \; x \in (-\infty, -2)$$

Q8.

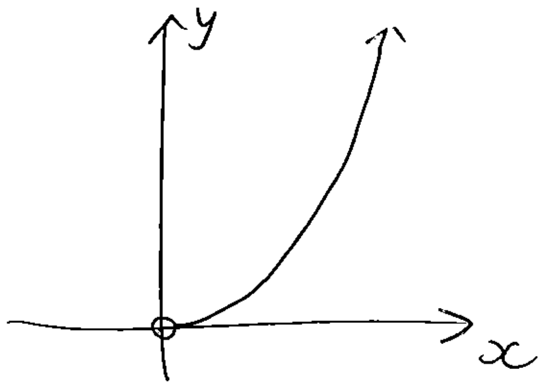
(a) $h(f(x)) = e^{2\log_e x + 3}$

$= e^3 . e^{2\log_e x}$

$= e^3 . e^{\log_e x^2}$

$= e^3 x^2, \quad x > 0$

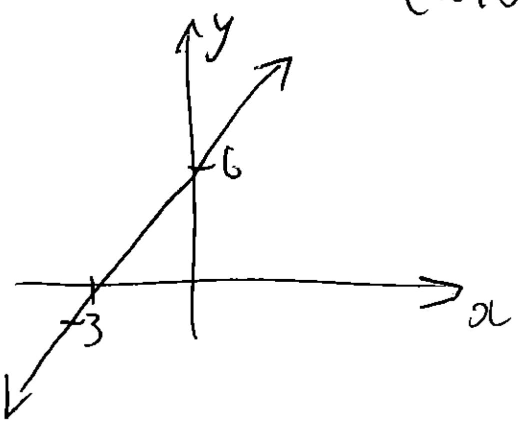
(b)



(c) $f(h(x)) = 2\log_e\left(e^{x+3}\right)$

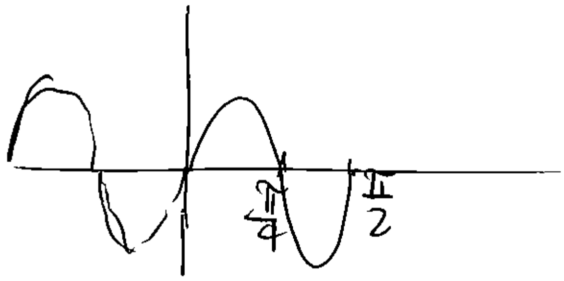
$= 2(x+3)$

(d)

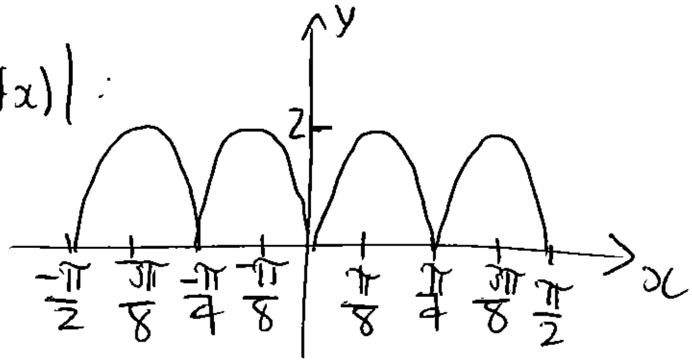


Q9. (a)

$y = 2\sin(4x)$



$\therefore$ Graph of $y = |2\sin(4x)|$:



(b)

$h(g(x))$ is defined if $\text{ran}(g) \subseteq \text{dom}(h)$

$\text{ran}(g) = [0, 2]$

$\text{dom}(h) = (-1, \infty)$

Since $\text{ran}(g) \subseteq \text{dom}(h)$, $\therefore$ $h(g(x))$ is defined.

(c) $h\left(g\left(\frac{7\pi}{24}\right)\right)$:

$g\left(\frac{7\pi}{24}\right) = \left|2\sin\left(4 \times \frac{7\pi}{24}\right)\right| = \left|2\sin\left(\frac{7\pi}{6}\right)\right|$

$= \left|2 \times \frac{-1}{2}\right| = |-1| = 1$

$\therefore h\left(g\left(\frac{7\pi}{24}\right)\right) = h(1) = \log_e 2$

Q10. (a) $2x-5>0$

$\therefore x>\frac{5}{2}$

$\therefore D=\left(\frac{5}{2},\infty\right)$

dom(h)	ran(h)
$\left(\frac{5}{2},\infty\right)$	$\mathbb{R}$
dom(h^{-1})	ran(h^{-1})
$\mathbb{R}$	$\left(\frac{5}{2},\infty\right)$

(b) $y=\frac{1}{2}\log_e(2x-5)$

$\downarrow$

$x=\frac{1}{2}\log_e(2y-5)$

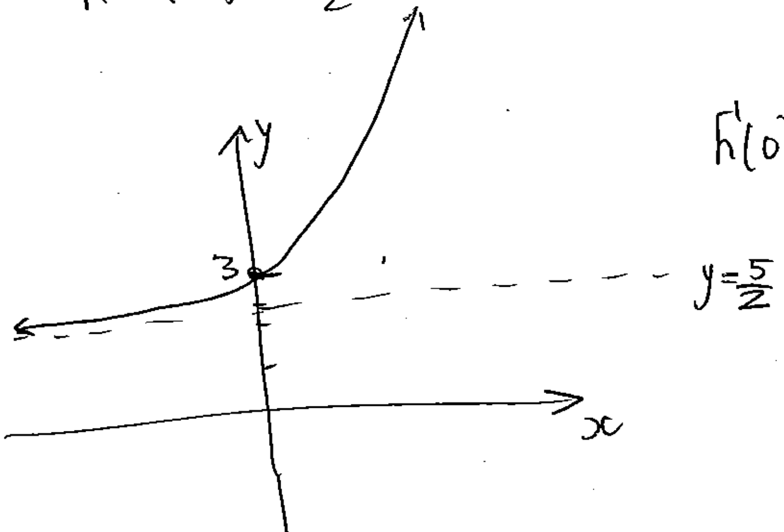
$2x=\log_e(2y-5)$

$\therefore e^{2x}=2y-5$

$y=\frac{1}{2}e^{2x}+\frac{5}{2}$

$\therefore h^{-1}(x)=\frac{1}{2}e^{2x}+\frac{5}{2},\ x\in\mathbb{R}$

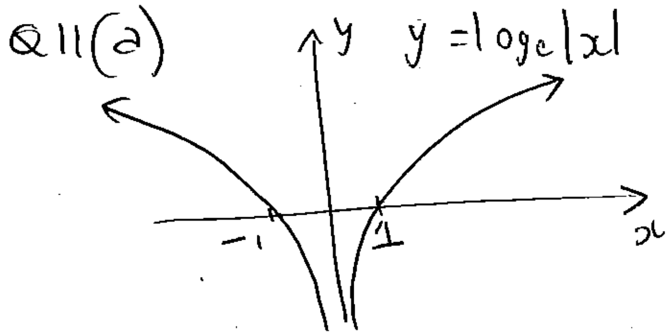
(c)



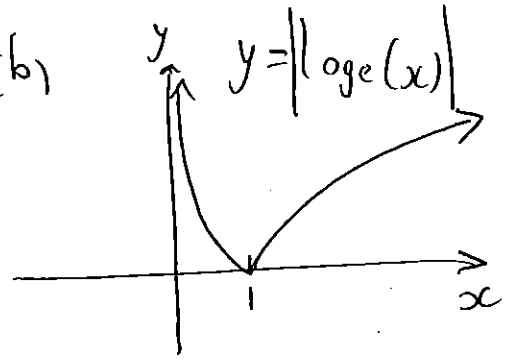
$h^{-1}(0)=\frac{1}{2}e^{0}+\frac{5}{2}=3$

$\therefore(0,3)$

Q11(a)

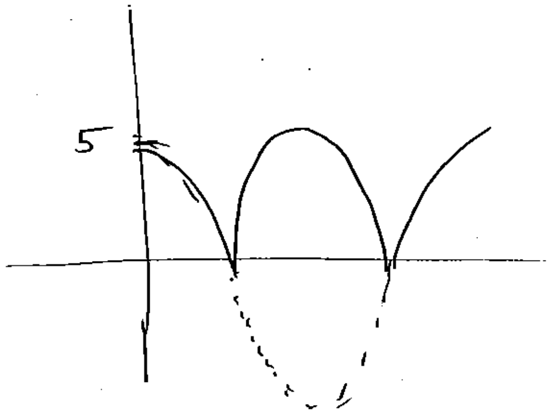


(b)

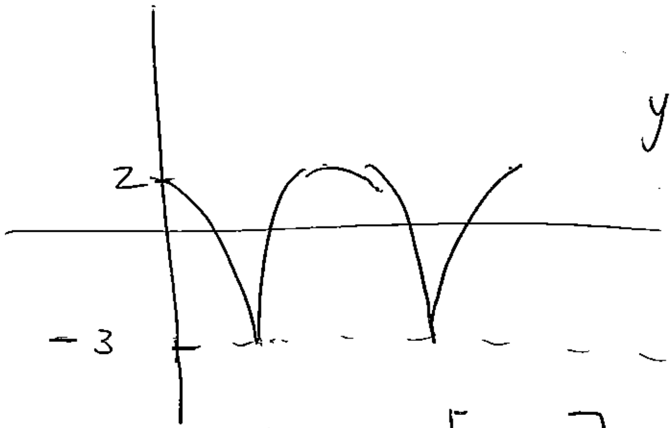


Q12.

$$y = 5|\cos(6x)| - 3$$



$$y = 5|\cos(6x)|$$



$$y = 5|\cos(6x)| - 3$$

Range: $[-3, 2]$