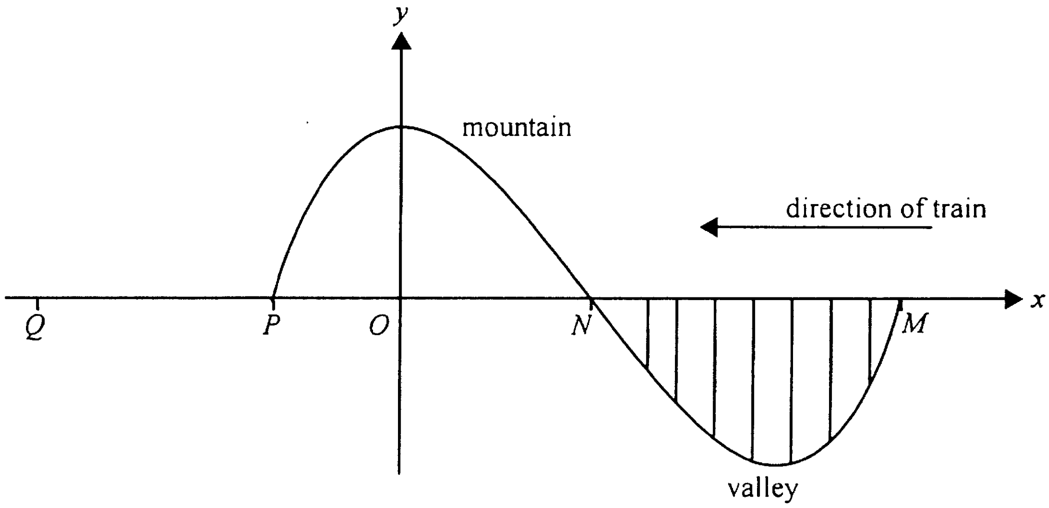
Question 2



A train is travelling at a constant speed of w km/h along a straight level track from M towards Q. The train will travel along a section of track $MNPQ$.

Section MN passes along a bridge over a valley.
Section NP passes through a tunnel in a mountain.
Section PQ is 6.2 km long.

From M to P, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph of

$$y = \frac{1}{200}(ax^3 + bx^2 + c) \text{ where } a, b \text{ and } c \text{ are real numbers.}$$

All measurements are in kilometres.

a. The curve defined from M to P passes through $N(2, 0)$. The gradient of the curve at N is –0.06 and the curve has a turning point at $x = 4$.

i. From this information write down three simultaneous equations in a, b and c.

$f(2) = 0$

$\therefore 0 = \frac{1}{200}\left(a(2)^3 + b(2^2) + c\right)$

$\therefore 0 = 8a + 4b + c \quad (1)$

$f'(2) = -0.06, \quad f'(4) = 0$

$f'(x) = \frac{1}{200}(3ax^2 + 2bx)$

$\therefore 0 = \frac{1}{200}(3a \times 4^2 + 2b(4))$

$\therefore 0 = 48a + 8b$

$\therefore 0 = 6a + b \quad (2)$

$-0.06 = \frac{1}{200}(3a \times 2^2 + 2b(2))$

$\therefore -12 = 12a + 4b$

$\therefore -3 = 3a + b \quad (3)$

ii. Hence show that $a = 1$, $b = -6$ and $c = 16$.

$0 = 6a + b$ (A)
$-3 = 3a + b$ (B)
$0 = 8a + 4b + c$ (C)

(A) − (B): $3 = 3a$ ∴ $a = 1$
Substitute for a in (A):
$b = -6$

Substitute for a, b in (C):
$0 = 8 \times 1 + 4 \times -6 + c$
∴ $0 = 8 - 24 + c$
∴ $c = 16$

3 + 2 = 5 marks

b. Find, giving exact values

i. the coordinates of M and P

$y = \frac{1}{200}(x^3 - 6x^2 + 16)$ At M, P, $y = 0$
∴ $0 = x^3 - 6x^2 + 16$

ii. the length of the tunnel

∴ $M = (2\sqrt{3} + 2, 0)$, $P = (2 - 2\sqrt{3}, 0)$

Length of tunnel
$= 2 - (2 - 2\sqrt{3})$
$= 2\sqrt{3}$ km

iii. the maximum depth of the valley below the train track.

For t/p, $\frac{dy}{dx} = 0$ ∴ $\frac{1}{200}(3x^2 - 12x) = 0$
∴ $3x^2 - 12x = 0$
$3x(x - 4) = 0$
∴ $x = 0, 4$

2 + 1 + 1 = 4 marks

If $x = 4$, $y = \frac{1}{200}(4^3 - 6 \times 4^2 + 16)$
$= \frac{-2}{25}$

∴ Maximum depth $= \frac{2}{25}$ km below the track

The driver sees a large rock on the track at a point Q, 6.2 km from P. The driver puts on the brakes at the instant that the front of the train comes out of the tunnel at P.

From its initial speed of w km/h, the train slows down from point P so that its speed v km/h is given by

$$v = k\log_e\left(\frac{(d+1)}{7}\right),$$

where d km is the distance of the front of the train from P and k is a real constant.

c. Find the value of k in terms of w.

When $d=0$, $V=w$

$\therefore w = k\log_e\left(\frac{1}{7}\right)$

$k = \dfrac{w}{\log_e\left(\frac{1}{7}\right)}$

1 mark

d. If $v=\dfrac{120\log_e(2)}{\log_e(7)}$ when $d=2.5$, find the value of w.

$V = \dfrac{w}{\log_e\left(\frac{1}{7}\right)} \times \log_e\left(\dfrac{d+1}{7}\right)$

When $d=2.5$, $\dfrac{120\log_e(2)}{\log_e(7)} = \dfrac{w}{\log_e\left(\frac{1}{7}\right)} \times \log_e\left(\dfrac{3.5}{7}\right)$

$\therefore \dfrac{120\log_e 2}{\log_e(7)} = \dfrac{w\log_e\left(\frac{1}{2}\right)}{\log_e\left(\frac{1}{7}\right)}$

$\therefore \dfrac{120\log_e(2)}{\log_e(7)} = \dfrac{w\times -\log_e(2)}{-\log_e(7)}$

$\therefore w=120.$

2 marks

e. Find the exact distance from the front of the train to the large rock when the train finally stops.

$\therefore V = \dfrac{120}{\log_e\left(\frac{1}{7}\right)}\log_e\left(\dfrac{d+1}{7}\right)$

When $v=0$, $\log_e\left(\dfrac{d+1}{7}\right)=0$

$\therefore \dfrac{d+1}{7}=1$

$\therefore d=6.$

$\therefore$ The distance to rock $= 6.2-6 = 0.2$ km

2 marks

Total 14 marks