SOLUTIONS

Question 2

a. Find an antiderivative of $\frac{1}{3x-4}$ with respect to x.

$$\int \frac{1}{3x-4}\,dx = \frac{1}{3}\log_e(3x-4)$$

("+c" not required since it asks for <u>an</u> antiderivative)

1 mark

Question 2

Find an anti-derivative of $\frac{1}{(2x-1)^3}$ with respect to x.

$$\int (2x-1)^{-3}\,dx = \frac{(2x-1)^{-2}}{-2\times 2} + c$$

$$= -\frac{1}{4(2x-1)^2} + c$$

2 marks

("+c" not required since <u>an</u> antiderivative is asked for).

Question 7

If $f(x) = x\cos(3x)$, then $f'(x) = \cos(3x) - 3x\sin(3x)$.

Use this fact to find an antiderivative of $x\sin(3x)$.

$$\frac{d}{dx}\left(x\cos(3x)\right) = \cos(3x) - 3x\sin(3x)$$

$$\therefore x\cos 3x = \int \cos 3x\,dx - \int 3x\sin 3x\,dx$$

$$\therefore x\cos 3x - \int \cos 3x\,dx = -\int 3x\sin 3x\,dx$$

$$\therefore \int \cos 3x\,dx - x\cos 3x = 3\int x\sin 3x\,dx$$

$$\therefore \frac{1}{3}\sin 3x - x\cos 3x = 3\int x\sin 3x\,dx$$

$$\therefore \frac{1}{9}\sin 3x - \frac{1}{3}x\cos 3x = \int x\sin 3x\,dx$$

3 marks

$$\therefore \int x\sin 3x\,dx = \frac{1}{9}\sin 3x - \frac{1}{3}x\cos 3x$$

("+c" not required since an antiderivative was asked for)

Question 1

a. Let $y = (3x^2 - 5x)^5$. Find $\frac{dy}{dx}$.

(Use Chain Rule)

$$y = (3x^2 - 5x)^5$$

$$\therefore \frac{dy}{dx} = 5(3x^2 - 5x)^4 \times (6x - 5)$$

$$\therefore \frac{dy}{dx} = 5(6x - 5)(3x^2 - 5x)^4$$

b. Let $f(x) = xe^{3x}$. Evaluate $f'(0)$.

(Use Product Rule)

$$y = xe^{3x}$$

$$\therefore \frac{dy}{dx} = e^{3x} + x \times 3e^{3x}$$

$$\frac{dy}{dx} = e^{3x} + 3xe^{3x}$$

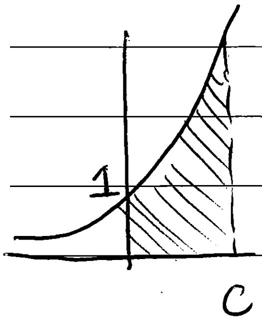
$$\therefore f'(0) = e^0 + 3 \times 0 \times e^0$$

$$= 1$$

2 + 3 = 5 marks

Question 5

The area of the region bounded by the y-axis, the x-axis, the curve $y = e^{2x}$ and the line $x = C$, where C is a positive real constant, is $\frac{5}{2}$. Find C.



Required area $= \int_0^C e^{2x}\,dx$

$\therefore \frac{5}{2} = \int_0^C e^{2x}\,dx$

$\therefore \left[\frac{1}{2}e^{2x}\right]_0^C = \frac{5}{2}$

$\therefore \frac{1}{2}e^{2c} - \frac{1}{2}e^{0} = \frac{5}{2}$

$\therefore \frac{1}{2}e^{2c} = 3$

$e^{2c} = 6 \quad \therefore \quad 2c = \log_e 6$

$\therefore c = \frac{1}{2}\log_e 6.$

3 marks

Question 1

Let $f(x) = \frac{x^3}{\sin(x)}$. Find $f'(x)$.

$f(x) = \frac{x^3}{\sin x}$

Use Quotient Rule: $\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$, $u = x^3$, $v = \sin x$

$v = \sin x \quad u = x^3$

$v' = \cos x \quad u' = 3x^2$

2 marks

$\therefore f'(x) = \frac{3x^2 \sin x - x^3 \cos x}{\sin^2 x}$

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

a. Differentiate $\sqrt{4-x}$ with respect to x.

$$y = (4-x)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \times (4-x)^{-\frac{1}{2}} \times -1$$

$$= \frac{-1}{2\sqrt{4-x}}$$

1 mark

b. If $g(x) = x^2\sin(2x)$, find $g'\left(\frac{\pi}{6}\right)$.

Let $u = x^2$, $u' = 2x$

$v = \sin(2x)$, $v' = 2\cos(2x)$

Use Product Rule:

$$g'(x) = 2x\sin(2x) + 2x^2\cos(2x)$$

$$g'\left(\frac{\pi}{6}\right) = 2 \times \frac{\pi}{6}\sin\left(\frac{\pi}{3}\right) + 2\left(\frac{\pi}{6}\right)^2\cos\left(\frac{\pi}{3}\right)$$

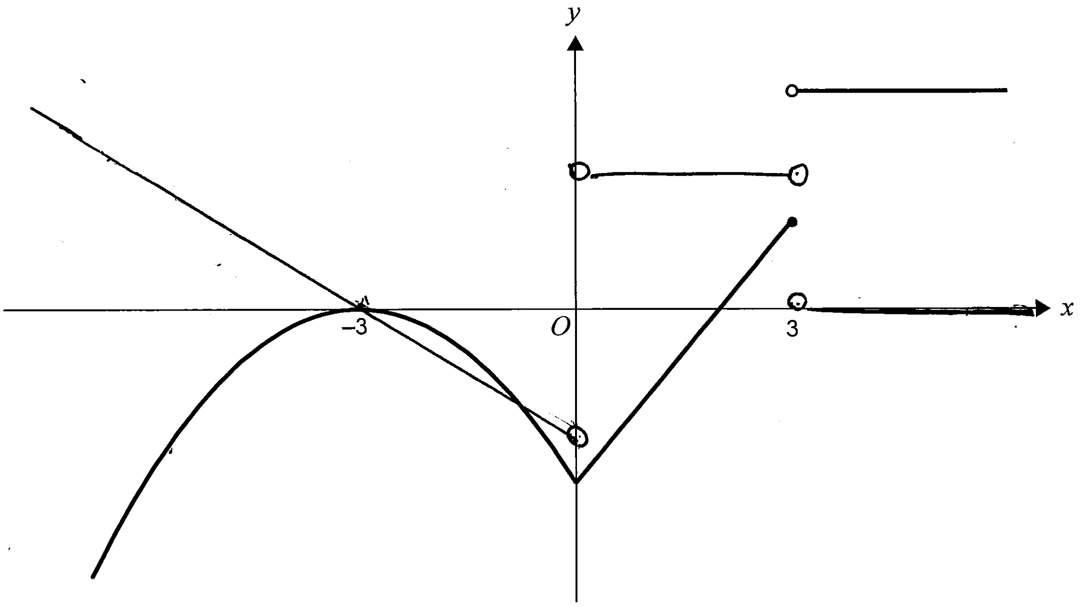
$$= \frac{\pi}{3} \times \frac{\sqrt{3}}{2} + 2 \times \frac{\pi^2}{36} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}\pi}{6} + \frac{\pi^2}{36}$$

2 marks

Question 3

The diagram shows the graph of a function with domain R.



a. For the graph shown above, sketch on the same set of axes the graph of the derivative function.

3 marks

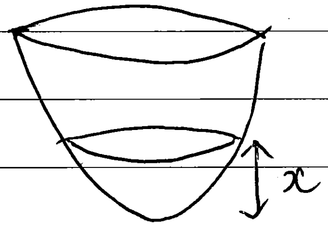
b. Write down the domain of the derivative function.

$R \setminus \{0, 3\}$

1 mark

Question 4

A wine glass is being filled with wine at a rate of $8\ \text{cm}^3/\text{s}$. The volume, $V\ \text{cm}^3$, of wine in the glass when the depth of wine in the glass is x cm is given by $V = 4x^{\frac{3}{2}}$. Find the rate at which the depth of wine in the glass is increasing when the depth is 4 cm.



Variables: V, x, t

Find: $\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{dx}{dV} \cdot \frac{dV}{dt}$

$V = 4x^{3/2}$

$\therefore \frac{dV}{dx} = 4 \times \frac{3}{2} x^{\frac{1}{2}}$

$= 6\sqrt{x}$

$\therefore \frac{dx}{dV} = \frac{1}{6\sqrt{x}}$

$\therefore \frac{dx}{dt} = \frac{1}{6\sqrt{x}} \times 8$

$\frac{dx}{dt} = \frac{8}{6\sqrt{x}}$

When $x = 4$,

$\frac{dx}{dt} = \frac{8}{6 \times \sqrt{4}} = \frac{8}{12} = \frac{2}{3}$ cm/sec

Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

a. If $y = (x^2 - 5x)^4$, find $\frac{dy}{dx}$.

$y = (x^2 - 5x)^4$

$\frac{dy}{dx} = 4 \times (x^2 - 5x)^3 \times (2x - 5)$

$= 4(2x - 5)(x^2 - 5x)^3$

1 mark

b. If $f(x) = \frac{x}{\sin(x)}$, find $f'\left(\frac{\pi}{2}\right)$.

$u = x \quad v = \sin x$

$y = \frac{u}{v} \quad u' = 1 \quad v' = \cos x$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$\therefore \frac{dy}{dx} = \frac{\sin x - x\cos x}{(\sin x)^2}$

2 marks

Question 10

Let $f: R \rightarrow R$, $f(x) = e^{-mx} + 3x$, where m is a positive rational number.

a. **i.** Find, in terms of m, the x-coordinate of the stationary point of the graph of $y = f(x)$.

$f'(x) = -me^{-mx} + 3$

For a stationary point, $f'(x) = 0$

$\therefore 3 - me^{-mx} = 0$

$\therefore me^{-mx} = 3$

$\therefore e^{-mx} = \frac{3}{m}$ $\therefore -mx = \log_e\left(\frac{3}{m}\right)$ $x = \frac{-1}{m}\log_e\left(\frac{3}{m}\right)$

ii. State the values of m such that the x-coordinate of this stationary point is a positive number.

$x = \frac{-1}{m}\log_e\left(\frac{3}{m}\right)$ Since $m > 0$, $\frac{1}{m}$ is positive.

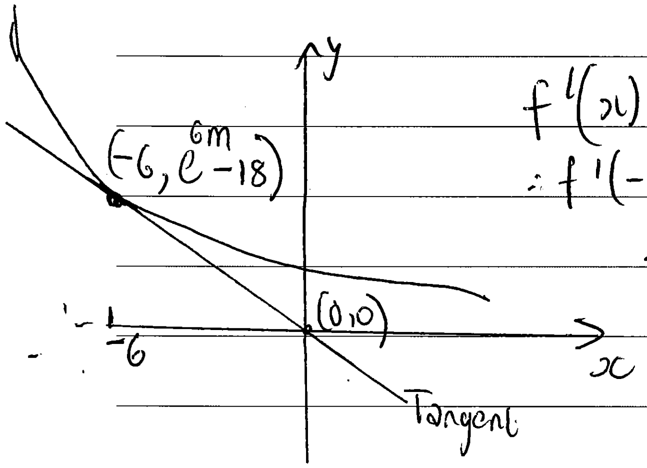
$\therefore x = \frac{1}{m}\log_e\left(\frac{m}{3}\right)$ $\therefore x > 0$ if $\log_e\left(\frac{m}{3}\right) > 0$

$\log_e\left(\frac{m}{3}\right) > 0$ if $\frac{m}{3} > 1$ $\therefore \{m : m > 3\}$

2 + 1 = 3 marks

b. For a particular value of m, the tangent to the graph of $y = f(x)$ at $x = -6$ passes through the origin.

Find this value of m.



$f'(x) = -me^{-mx} + 3$

$\therefore f'(-6) = -me^{6m} + 3$

$\therefore$ Gradient of tangent at $x = -6$ is equal to $3 - me^{6m}$

But gradient of tangent is also equal to:

$\frac{e^{6m} - 18}{-6 - 0} = \frac{e^{6m} - 18}{-6}$

3 marks

$\therefore 3 - me^{6m} = \frac{e^{6m} - 18}{-6}$

$\therefore -18 + 6me^{6m} = e^{6m} - 18$

$\therefore 6me^{6m} = e^{6m}$

$\therefore 6m = 1$ $m = \frac{1}{6}$

END OF QUESTION AND ANSWER BOOK