SOLUTIONS for TRIAL EXAM 1

Q1.

$$mx + 2y = 6$$
$$x + (m-1)y = -3$$

$$\begin{bmatrix} m & 2 \\ 1 & m-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

For no unique solution, $\det(A) = 0$

$$\therefore \quad m(m-1) - 2 = 0$$
$$m^2 - m - 2 = 0$$
$$(m-2)(m+1) = 0$$
$$m = 2, -1$$

If $m = 2$: $\quad 2x + 2y = 6 \Leftrightarrow x + y = 3$

$x + y = -3 \Leftrightarrow x + y = -3$

These are parallel lines with no unique solution

If $m = -1$:

$-x + 2y = 6 \Leftrightarrow x - 2y = -6$

$x - 2y = -3 \Leftrightarrow x - 2y = -3$

These are parallel lines with no unique solution

$$\therefore \quad m = -1, 2$$

Q2.

$$2\sin^2(2x) - 1 = 0$$

$$\therefore \sin^2(2x) = \frac{1}{2}$$

$$\therefore \sin(2x) = \pm\frac{1}{\sqrt{2}}$$

$$\sin(2x) = \frac{1}{\sqrt{2}}, \quad 0 \le x \le \pi$$

$$\therefore \sin(2x) = \frac{1}{\sqrt{2}}, \quad 0 \le 2x \le 2\pi$$

$$\therefore 2x = \frac{\pi}{4}, \frac{3\pi}{4} \qquad \therefore x = \frac{\pi}{8}, \frac{3\pi}{8}$$

$$\sin(2x) = -\frac{1}{\sqrt{2}}, \quad 0 \le 2x \le 2\pi$$

$$2x = \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\therefore x = \frac{5\pi}{8}, \frac{7\pi}{8}$$

$$\therefore x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$

Q3. $2\log_2(2x+1) + \log_2 3 = 3$

$\therefore \log_2((2x+1)^2) + \log_2 3 = 3$

$\log_2(3(2x+1)^2) = 3$

$3(2x+1)^2 = 2^3$

$3(2x+1)^2 = 8$

$3(4x^2+4x+1) = 8$

$12x^2+12x+3 = 8$

$12x^2+12x-5 = 0$

$x = \dfrac{-12 \pm \sqrt{144+20\times 12}}{24}$

$x = \dfrac{-12 \pm \sqrt{384}}{24}$

$x = \dfrac{-12 \pm 8\sqrt{6}}{24}$

$x = \dfrac{-3 \pm 2\sqrt{6}}{6}$

$\therefore x = -\frac{1}{2} + \frac{\sqrt{6}}{3}$ or $x = -\frac{1}{2} - \frac{\sqrt{6}}{3}$

NB: Domain for solution:

$2x+1 > 0$

$\therefore x > -\frac{1}{2}$

$\sqrt{384} = \sqrt{16\times 24}$

$= 4\sqrt{24}$

$= 4\sqrt{4}\sqrt{6}$

$= 8\sqrt{6}$

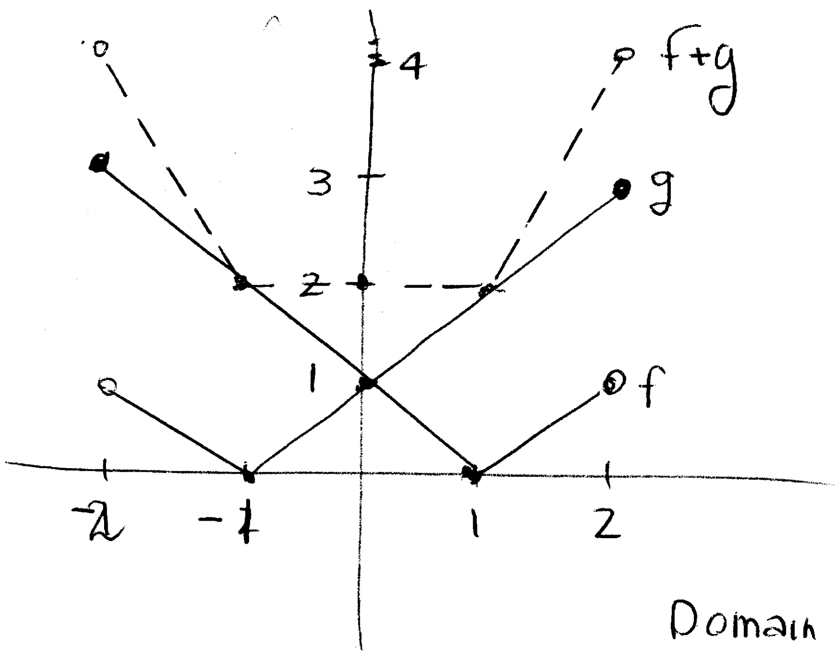
But since the domain of our solution is $x > -\frac{1}{2}$, only $x = -\frac{1}{2} + \frac{\sqrt{6}}{3}$ is a valid solution

$\therefore x = -\frac{1}{2} + \frac{\sqrt{6}}{3}$

Q4. (d)

$f:[-2,2) \to \mathbb{R},\ f(x)=|x-1|$

$g:(-2,2] \to \mathbb{R},\ g(x)=|x+1|$



Domain of $f+g$ is $(-2,2)$

Q5. (a)

$f(x) = kx^2 \tan(2x)$

$f'(x) = uv' + vu'$

$u = kx^2 \quad v = \tan(2x)$

$u' = 2kx \quad v' = 2\sec^2 2x = \dfrac{2}{\cos^2 2x}$

$f'(x) = kx^2 \cdot \dfrac{2}{\cos^2 2x} + 2kx\tan 2x$

$f'(x) = \dfrac{2kx^2}{\cos^2 2x} + 2kx\tan 2x$

$f'\left(\frac{\pi}{8}\right) = \dfrac{2k\left(\frac{\pi}{8}\right)^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + 2k \times \frac{\pi}{8} \times \tan\left(\frac{\pi}{4}\right)$

$= \dfrac{4k \times \pi^2}{64} + \dfrac{k\pi}{4}$

Q5 (Cont)

$$= \frac{k\pi^2}{16} + \frac{k\pi}{4}$$

$$= \frac{k\pi}{16}(4+\pi)$$

Q5.(c) $f'\left(\frac{\pi}{8}\right) = \frac{3}{4}$

$$\therefore \frac{3}{4} = \frac{k\pi}{16}(\pi+4)$$

$$\therefore 12 = k\pi(\pi+4)$$

$$k = \frac{12}{\pi(\pi+4)}$$

Q6.

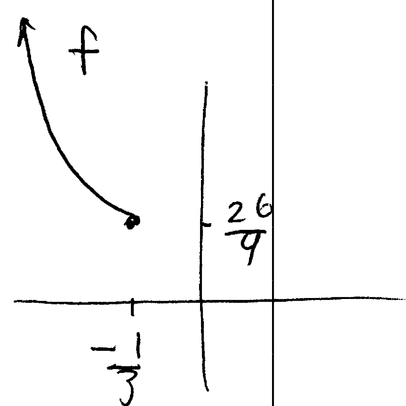
(a) $f: (-\infty, A] \to \mathbb{R},\ f(x) = x^2 + \frac{2x}{3} + 3$

$$\therefore y = x^2 + \frac{2x}{3} + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 + 3$$

$$y = \left(x + \frac{1}{3}\right)^2 - \frac{1}{9} + \frac{27}{9}$$

$$y = \left(x + \frac{1}{3}\right)^2 + \frac{26}{9}$$

$$\therefore A = -\frac{1}{3}$$



(b)

$$x = \left(y + \frac{1}{3}\right)^2 + \frac{26}{9}$$

$$x - \frac{26}{9} = \left(y + \frac{1}{3}\right)^2$$

$$y + \frac{1}{3} = \pm\sqrt{x - \frac{26}{9}}$$

$$y = -\frac{1}{3} \pm \sqrt{x - \frac{26}{9}}$$

dom(f)	ran(f)
$\left(-\infty, -\frac{1}{3}\right]$	$\left[\frac{26}{9}, \infty\right)$
dom(f^{-1})	ran(f^{-1})
$\left[\frac{26}{9}, \infty\right)$	$\left(-\infty, -\frac{1}{3}\right]$

But since ran $(f^{-1}) = \left(-\infty, -\frac{1}{3}\right]$

we must take the $-$ve square root

$$\therefore f^{-1}(x) = -\frac{1}{3} - \sqrt{x - \frac{26}{9}},\quad x \in \left[\frac{26}{9}, \infty\right)$$

Q7. (a) $f(x) = e^{3x+2}$

$f'(x) = 3e^{3x+2}$

$f'(0) = 3e^{2}$

$\therefore$ For gradient of normal at $x = 0$:

$m_n = -\dfrac{1}{3e^2}$

At $x = 0$, $f(0) = e^2$

$\therefore$ To find equation of normal:

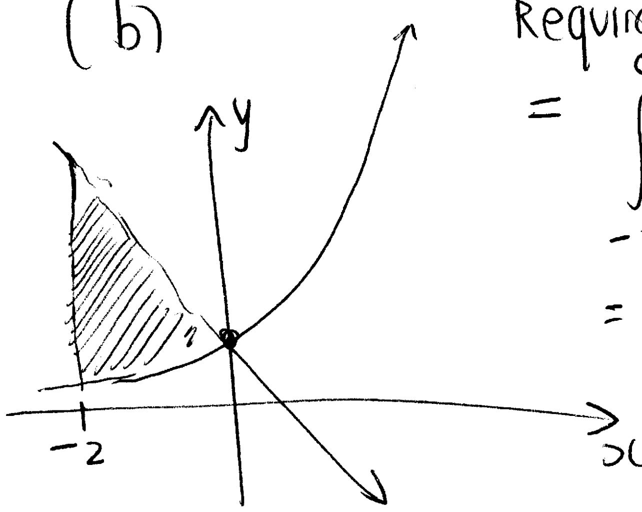
$y - y_1 = m(x - x_1)$ where $m = -\dfrac{3}{e^2}$

$(x_1, y_1) = (0, e^2)$

$\therefore y - e^2 = \dfrac{-1}{3e^2}(x - 0)$

$y = \dfrac{-x}{3e^2} + e^2$

(b)



Required area

$$= \int_{-2}^{0} \left(\frac{-x}{3e^2} + e^2\right) - e^{3x+2}\, dx$$

$$= \left[\frac{-x^2}{6e^2} + e^2 x - \frac{1}{3}e^{3x+2}\right]_{-2}^{0}$$

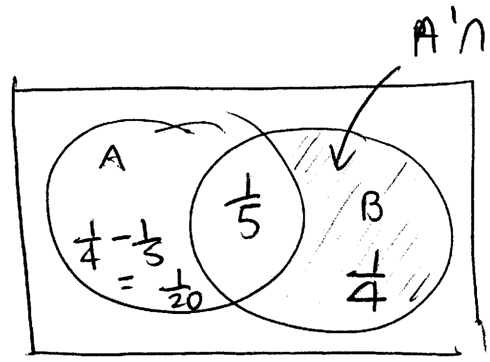
$$= -\frac{1}{3}e^2 - \left(\frac{-4}{6e^2} - 2e^2 - \frac{1}{3}e^{-4}\right)$$

$$= \frac{-1}{3}e^2 + \frac{2}{3}e^2 + 2e^2 - \frac{1}{3\cancel{e}}e^{-4}$$

$$= -\frac{1}{3}e^2 + \frac{2}{3}e^2 + 2e^2 - \frac{1}{3e^4}$$

$$= \frac{7}{3}e^2 - \frac{1}{3e^4} \text{ sq. units.}$$

Q8.



(a) From Venn Diagram, $Pr(B) = \frac{1}{5} + \frac{1}{4}$

$$= \frac{9}{20}$$

(b)

If A, B are independent, then:

$Pr(A) \cdot Pr(B) = Pr(A \cap B)$

$\therefore \frac{1}{4} \times Pr(B) = \frac{1}{5}$

$\therefore Pr(B) = \frac{4}{5}$

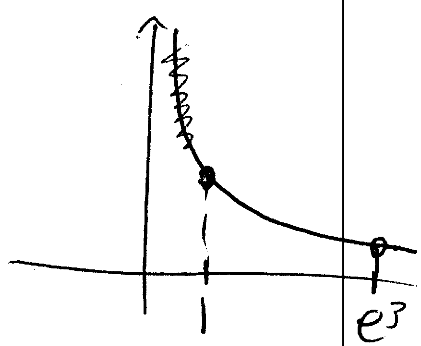
$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

$\therefore Pr(A \cup B) = \frac{1}{4} + \frac{4}{5} - \frac{1}{5}$

$= \frac{1}{4} + \frac{3}{5} = \frac{5}{20} + \frac{12}{20} = \frac{17}{20}$

Q9.

(a) $f(x) = \begin{cases} \frac{k}{2x}, & 1 \le x \le e^3 \\ 0, & \text{elsewhere} \end{cases}$



$$\int_1^{e^3} \frac{k}{2x}\, dx = 1$$

$$\therefore \left[\frac{k}{2}\log_e x\right]_1^{e^3} = 1$$

$$\therefore \frac{k}{2}\log_e e^3 - \frac{k}{2}\log_e 1 = 1$$

$$\therefore \frac{k}{2} \times 3\log_e e - 0 = 1$$

$$\therefore \frac{3k}{2} = 1$$

$$\therefore k = \frac{2}{3}$$

(b) $\Pr(X<10 \mid X \geq 1) = \dfrac{\Pr(X<10 \cap X \geq 1)}{\Pr(X \geq 1)}$

$$= \frac{\Pr(1 \le X < 10)}{\Pr(X \geq 1)}$$

$$= \frac{\Pr(1 \le X < 10)}{1}$$

$$\Pr(1 \le X < 10) = \int_1^{10} \frac{1}{2} \times \frac{2}{3} \cancel{\log} \frac{1}{x}\, dx = \left[\frac{1}{3}\log_e x\right]_1^{10}$$

$$= \frac{1}{3}\log_e 10$$

$\therefore$ Required probability $= \frac{1}{3}\log_e 10$.

Q9.

(c) $E(X) = \int_1^{e^3} x \times \frac{1}{3x}\, dx$

$= \int_1^{e^3} \frac{1}{3}\, dx$

$= \left[\frac{x}{3}\right]_1^{e^3}$

$= \frac{e^3}{3} - \frac{1}{3}$

$\therefore E(X) = \frac{1}{3}(e^3 - 1).$

Q10.

(a) Let $y = \frac{5}{2x^2-1}$

$\therefore y = \frac{5}{u}$ where $u = 2x^2 - 1$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= -\frac{5}{u^2} \times 4x$

$= \frac{-20x}{(2x^2-1)^2}$

Q10 (a) [(cont)

$$\therefore \frac{d}{dx}\left(\frac{5}{2x^2-1}\right) = \frac{-20x}{(2x^2-1)^2}$$

(b) To find: a where

$$\int_1^a \left(\frac{20x}{(2x^2-1)^2} + 1\right) dx = \frac{37}{7}$$

We must first find:

$$\int \frac{20x}{(2x^2-1)^2} + 1 \, dx$$

We know: $\frac{d}{dx}\left(\frac{5}{2x^2-1}\right) = \frac{-20x}{(2x^2-1)^2}$

$$\therefore \frac{5}{2x^2-1} = \int \frac{-20x}{(2x^2-1)^2} dx$$

$$\therefore \frac{-5}{2x^2-1} = \int \frac{20x}{(2x^2-1)^2} dx$$

$$\therefore \int \frac{20x}{(2x^2-1)^2} + 1 \, dx = \frac{-5}{2x^2-1} + x$$

$$\therefore \int_1^a \left(\frac{20x}{(2x^2-1)^2} + 1\right) dx = \left[\frac{-5}{2x^2-1} + x\right]_1^a$$

$$= \left(\frac{-5}{2a^2-1} + a\right) - \left(\frac{-5}{1} + 1\right)$$

$$= \frac{-5}{2a^2-1} + a + 4$$

Q 10 (b) [Cont]

$$\therefore \frac{-5}{2a^2-1} + a + 4 = \frac{37}{7}$$

$$\therefore \frac{-5}{2a^2-1} + a = \frac{9}{7}$$

If $a = 2$: $2 - \dfrac{5}{2\times 4 - 1}$

$$= 2 - \frac{5}{7}$$

$$= \frac{9}{7}$$

$\therefore a = 2.$