Question 3

Callum and Eloise are painting the office floor. Their office has an unusual shape, so they are trying to find a suitable model for the area of the floor, in order to calculate how much paint they need to buy. All dimensions are in metres unless specified otherwise.

a. They start by modelling it as a simple rectangle, with dimensions 2 m x 11.5 m. If the floor actually has area 16.95 m^2, what percentage error does this model have? Give your answer to two significant figures.

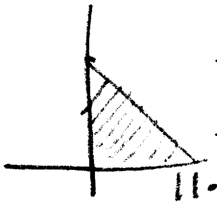
$2 \times 11.5 = 23$

Error $= 23 - 16.95 = 6.05$

% Error $= \frac{6.05}{16.95} \times 100 = 35.7\%$

2 marks

b. The second model they try is the area bounded by the x-axis and the curve $y = -\frac{3}{10}(x - 11.5)$. What is the area given by this model?



$\int_0^{11.5} -0.3(x - 11.5)\,dx = \frac{1587}{80}$ m^2

2 marks

c. They finally settle on a model that represents the area of the floor as the area between the two curves:

$$y = \frac{4}{5}\left(x^{\frac{5}{3}}\right)$$
$$y = 3\left(x^{\frac{3}{5}}\right)$$

i. Find all points of intersection. Simplify your answers as much as possible, and leave them in exact form.

$\frac{4}{5}x^{5/3} = 3x^{3/5}$

$4x^{5/3} = 15x^{3/5}$

$\cancel{x}$ $\left(4x^{5/3}\right)^{15} = \left(15x^{3/5}\right)^{15}$

$4^{15}x^{25} = (15)^{15}x^{9}$

If $x \neq 0$, $4^{15}x^{16} = 15^{15}$

$x^{16} = \left(\frac{15}{4}\right)^{15}$

$x = -\left(\frac{15}{4}\right)^{15/16}$ or $\left(\frac{15}{4}\right)^{15/16}$

Points are:

$\left(\left(\frac{15}{4}\right)^{15/16}, \frac{3 \times 15^{\frac{9}{16}}}{2^{9/8}}\right)$

and

$\left(-\left(\frac{15}{4}\right)^{15/16}, -\frac{3 \times 15^{9/16}}{2^{9/8}}\right)$

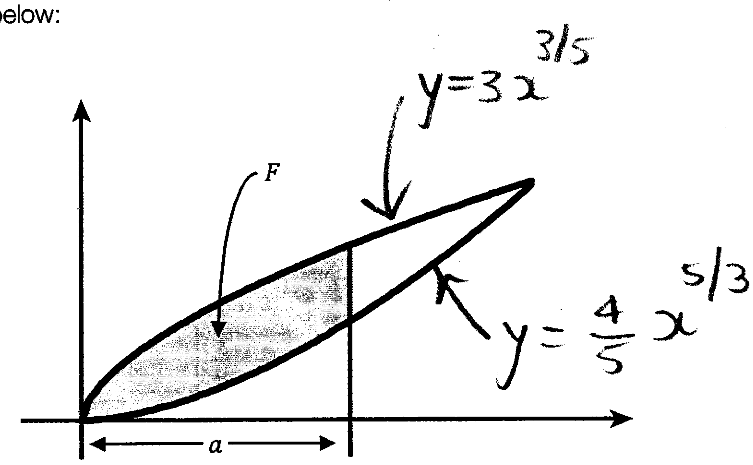
3 marks

ii. Write a definite integral, that, when evaluated, gives the area of the floor predicted by this model, and evaluate it to two decimal places.

$$\int_0^{\left(\frac{15}{4}\right)^{15/16}} \left(3x^{3/5} - \frac{4}{5}x^{5/3}\right)dx \simeq 5.45 \text{ m}^2$$

2 marks

Callum and Eloise realise that they can only afford 5 m² worth of paint. They decide to paint as much as they can, as shown below:



d. Write an expression for F, the area bounded by $x = 0$, $x = a$ and the two curves given in part c. above, as a definite integral in terms of a.

$$\int_0^a \left(3x^{3/5} - \frac{4}{5}x^{5/3}\right)dx$$

2 marks

e. Hence, find the value of a if they paint 5 m² of the floor. Give your answer to two decimal places.

$$\int_0^a \left(3x^{3/5} - \frac{4}{5}x^{5/3}\right)dx = 5$$

$$a \approx 2.74$$

1 mark

$\left(\text{since } 0 < a < \left(\frac{15}{4}\right)^{15/16}\right)$

Total 12 marks

Section B – Short-answer questions

Instructions

Answer all questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Questions

Question 1

Tiny Tony is building an open rectangular container with a square base so he can play with his rubber ducky. He wants the container to hold as much water as possible, but he only has $108cm^2$ of material to work with.

Let the dimensions of the container be $x\ cm \times x\ cm \times y\ cm$, and the volume and surface area of the box be V and S respectively.

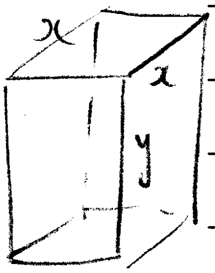
a. State any physical constraints on the values of x and y.

$0 < x$

$0 < y$

1 marks

b. Show that $V = 27x - \frac{1}{4}x^3$.



$x^2 + 4xy = 108$

$4xy = 108 - x^2$

$y = \frac{108 - x^2}{4x}$

$\therefore V = x^2 y$

$\therefore V = x^2 \left(\frac{108 - x^2}{4x}\right)$

$\therefore V = x\left(\frac{108 - x^2}{4}\right)$

2 marks

$\therefore V = 27x - \frac{x^3}{4}$

Domain: $x > 0$ and $y > 0$ $\therefore$ $108 - x^2 > 0$ $\therefore x < \sqrt{108}$

$\therefore$ Domain: $x \in (0, \sqrt{108})$

c. Find $\frac{dV}{dx}$ and hence the maximum possible volume of the container, stating its corresponding dimensions.

$$\frac{dV}{dx} = 27 - \frac{3x^2}{4}$$

For a stationary point, $V'(x) = 0$

$$\therefore 27 = \frac{3x^2}{4}$$

$$36 = x^2$$

$$x = \sqrt{36} \quad (\text{since } x \in (0, \sqrt{108}))$$

$\therefore x = 6$.

$V(6) = 27 \times 6 - \frac{6^3}{4} = 108$ Max volume $= 108$ cubic units

When $x = 6$, $y = \frac{108 - 6^2}{4 \times 6} = 3$ when $x = 6, y = 3$.

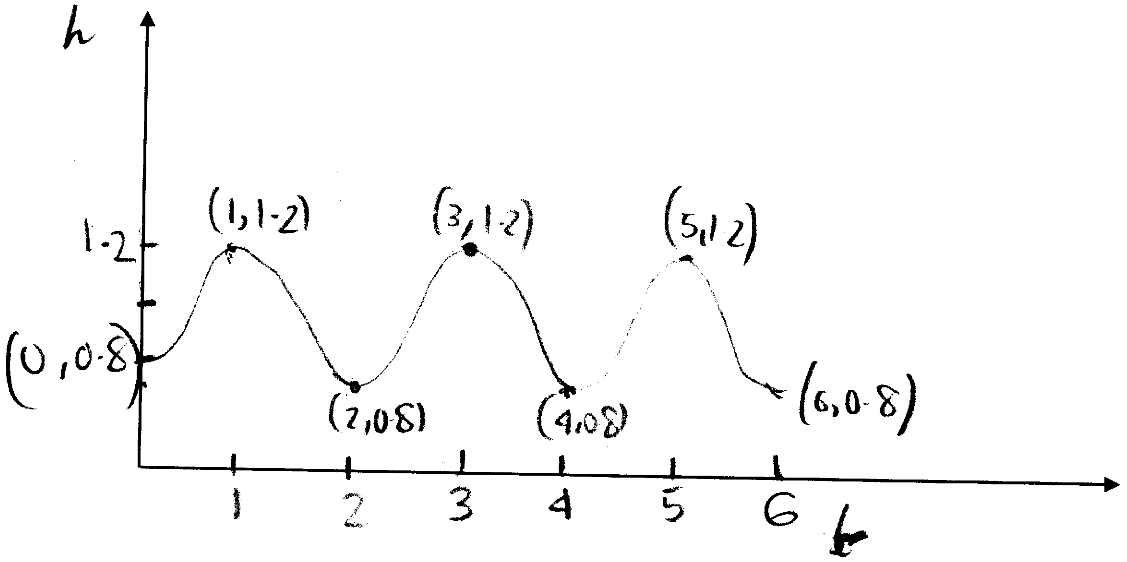
4 marks

d. Tony then fills the container with water up to a depth of $1cm$ and installs a wave machine along its base. He notices that the height of the water in the very center of the container, h, t seconds after turning on the wave machine appears to follow the curve:

$$h = \frac{\cos(\pi(t-3))}{5} + 1$$

Period $= \frac{2\pi}{\pi} = 2$

Sketch the graph of h against t on the axes below for $t \in [0,6]$. Label any stationary points.



4 marks

e. Tony wishes to drop his rubber ducky in to the center of the container, but he is afraid that if he does so while the water level is below $0.9cm$ he may hurt him. In first 6 seconds after turning on the wave machine, how much time is there during which it is not safe for Tony to drop his duck?

$$0.9 = 1 + \frac{\cos(\pi(t-3))}{5}, \quad 0 \le t \le 6$$

Solving:

$$t = \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{11}{3}, \frac{13}{3}, \frac{17}{3}$$

$h(t) < 0.9$ for

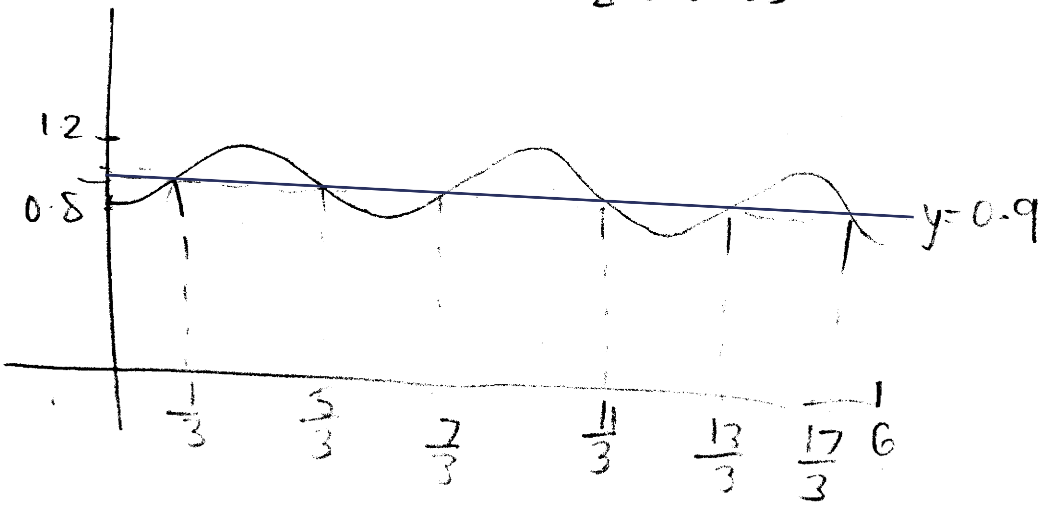
$$t \in \left[0, \tfrac{1}{3}\right) \cup \left(\tfrac{5}{3}, \tfrac{7}{3}\right) \cup \left(\tfrac{11}{3}, \tfrac{13}{3}\right) \cup \left(\tfrac{17}{3}, 6\right)$$

$\therefore$ Required time $= \frac{1}{3} + \frac{2}{3} + \frac{2}{3} + \frac{1}{3}$

$= 2$ seconds

4 marks

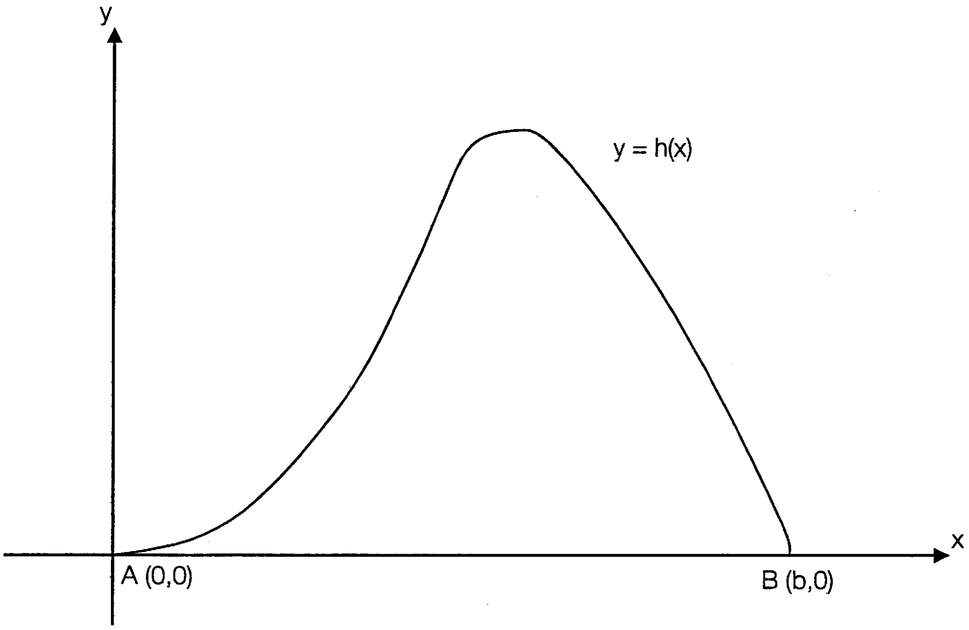
Total: 15 marks



Question 2

William the Conqueror has just won a battle at town A and now has to cross over a mountain to reach town B where he plans to have his next battle. The height of the mountain can be given by the equation:

$$h(x) = e^{\frac{-(6-x)^2}{40}}(-3x^3 + 36x^2),\ \ 0 \le x \le b$$



a.

i. Find the value of b, the horizontal distance from town A to town B.

$h(x) = 0 \quad \therefore x = 0, 12$

$b = 12$

1 mark

ii. State the derivative function of h(x).

$$\frac{dh}{dx} = \frac{3x}{20}\left(x^3 - 18x^2 + 112x + 480\right)e^{\frac{-x^2}{40} + \frac{3x}{10} - \frac{9}{10}}$$

iii. Find the maximum height of the mountain between town A and town B correct to 2 decimal places, include units in your answer.

For a stationary point, $h'(x) = 0$

$x = -4.37, 7.27141, 15.1002$

Since we require $0 < x < 12$,

$x = 7.27141$

$h(7.27141) = 720.343$

Max height $= 720.34$ m

2 marks

b. William's army also has to transport catapults to town B in order to guarantee victory. The maximum gradient a catapult can be pushed up is 150 m/km.

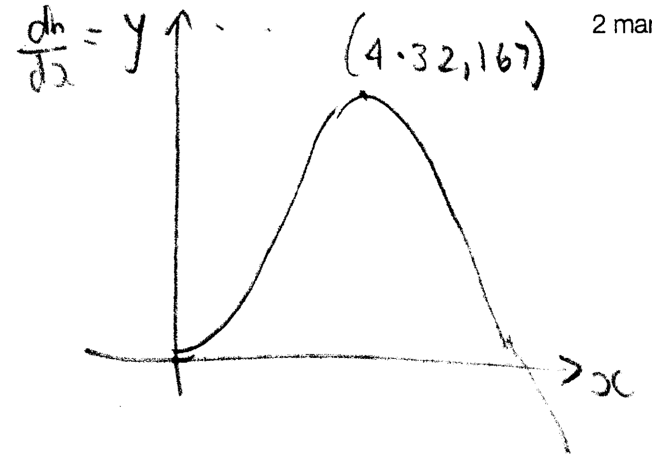
i. Will William be able to get his catapult from town A to town B?

Graph the derivative function $h'(x)$ (see below) for $0 < x < 12$

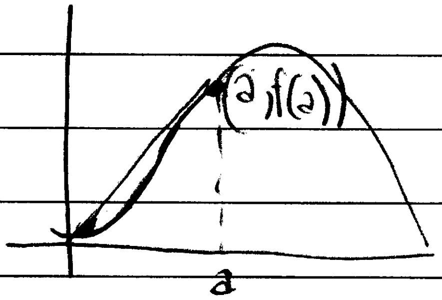
Clearly, $\frac{dh}{dx} > 150$ for values of x in the interval $(0, 12)$

William will not be able to get his catapult from A to B.

2 marks



ii. In order to make the climb easier for his army William decides to build a ramp up the mountain. This ramp starts at town A and has a constant rate of change of 100 m/km. What are the coordinates for the end position of the ramp correct to two decimal places?



$$\frac{h(a)-h(0)}{a-0} = 100$$

$$\therefore \frac{h(a)}{a} = 100$$

Solving for a: $a = 0, 4.79893, 7.20107$

Clearly, require $a = 4.7989$

$h(4.7989) \simeq 479.89$

$(4.80, 479.89)$

3 marks

c. Through experience in transporting armies, William also knows that his soldiers will be too tired to fight if they have to climb above 600m.

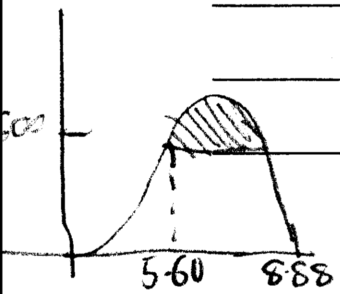
i. Find the horizontal distances at which the height of the mountain is 600m correct to two decimal places

Solve $h(x) = 600$ for x

$x = 5.60, 8.88$

2 marks

ii. Write a definite integral that evaluates the area of the graph h(x) above h(x) = 600 (do not worry about the units used)



$$\int_{5.60}^{8.88} \left(h(x) - 600\right) dx$$

1 mark

d. Finally William decides the best way to avoid having to climb over the mountain would be to build a tunnel through it. The maximum length of tunnel that William can build is 2km. What is the minimum height at which William can build this tunnel correct to two decimal places?

Solving $h(p) = h(p+2)$, $0 < p < 12$

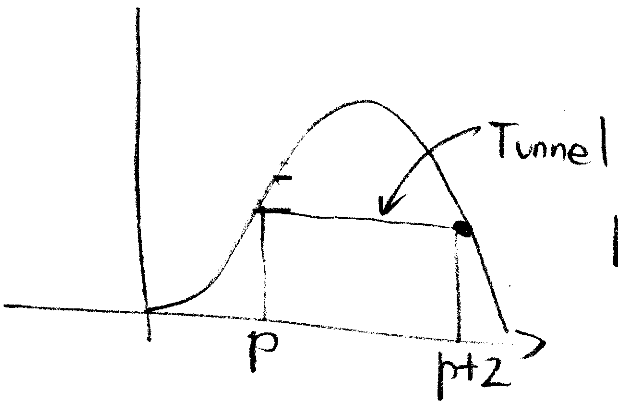
$p = 6.26043$

$h(6.26043) \approx 673.709$

$\therefore$ Minimum height $= 673.71$ m

2 marks

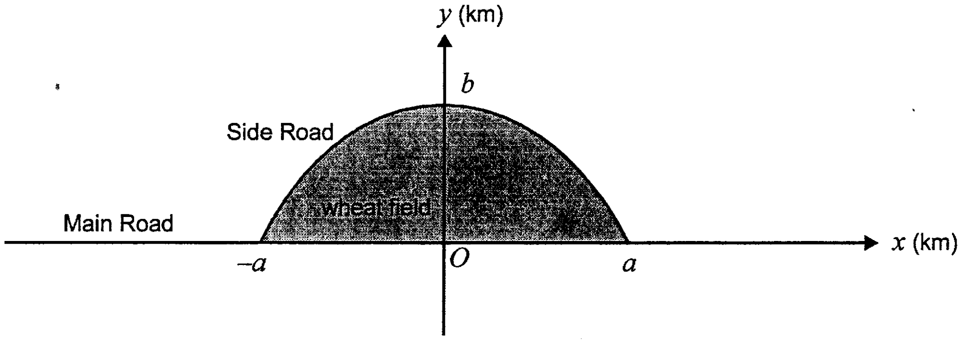
Total: 14 marks



$h(p) = h(p+2)$

Question 3

Tasmania Jones' wheat field lies between two roads as shown in the diagram below.
Main Road lies along the x-axis and Side Road lies along the curve with equation $y = 3 - e^x - e^{-x}$.



a. The y-axis intercept of the graph representing Side Road is b.
Show that $b = 1$.

Let $x = 0$

$y = 3 - e^0 - e^0$
$= 1$

1 mark

b. Find the exact value of a.

$3 - e^x - e^{-x} = 0$

Solving : $x = \log_e\left(\frac{3-\sqrt{5}}{2}\right)$

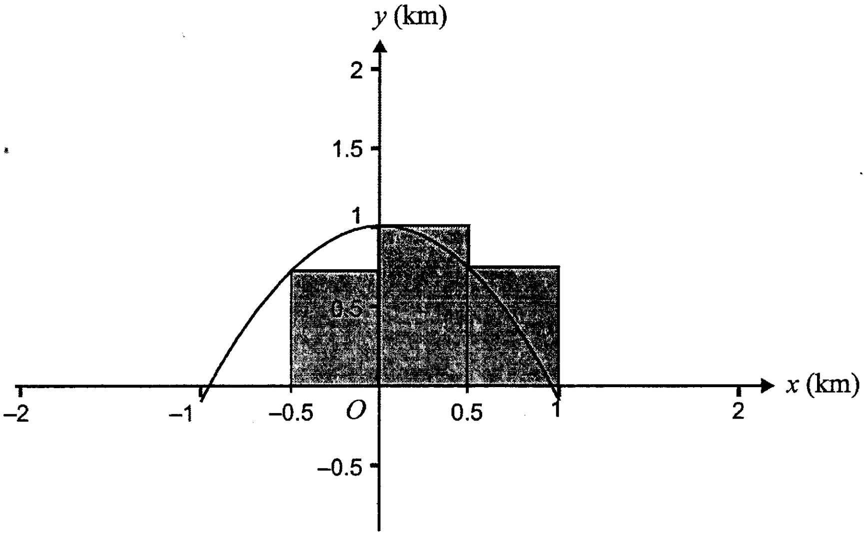
or $x = \log_e\left(\frac{3+\sqrt{5}}{2}\right)$

1 mark

$\therefore \quad a = \log_e\left(\frac{3+\sqrt{5}}{2}\right)$

OR : Use :
Solve$(3 - e^x - e^{-x} = 0, x) \mid 0 < x$

c. Since a is close to 1, Tasmania finds an approximation to the area of the wheat field by using rectangles of width 0.5 km, as shown on the following diagram.



i. Complete the table of values for y, where $y = 3 - e^x - e^{-x}$, giving values correct to two decimal places.

x	−0.5	0	0.5
y	0.74	1	0.74

ii. Use the table to find Tasmania's approximation to the area of the wheat field, measured in square kilometres, correct to one decimal place.

$0.5 \times 0.74 + 0.5 \times 1 + 0.5 \times 0.74$

$= 0.74 + 0.5$

$= 1.24$

$\simeq 1.2 \text{ km}^2$

iii. Tasmania uses this approximation to the area to estimate the value of the wheat in his field at harvest time. He estimates that he will obtain w kg of wheat from each square kilometre of field. The current price paid to growers is \$$m$ per kg of wheat. Write a formula for his estimated value, \$$V$, of the wheat in his field.

No. of kg of wheat $= 1.2w$

$V = 1.2wm$

1 + 2 + 1 = 4 marks

d. Tasmania Jones decides to find another approximation to the area of the wheat field. He approximates the curve representing Side Road with a parabola which passes through the points (0, 1), (1, 0) and (–1, 0). He finds the area enclosed by the parabola and the x-axis as an approximation to the area of his wheat field.

i. Find the equation of this parabola.

$$y = 1 - x^2$$

ii. Find the area enclosed by the parabola and the x-axis, giving your answer correct to two decimal places.

$$\int_{-1}^{1} (1 - x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3} \simeq 1.33 \text{ km}^2$$

1 + 2 = 3 marks

e. Find the values of k, where k is a positive real number, for which the equation $3 - ke^x - e^{-x} = 0$ has one or more solutions for x.

$$3 - ke^x - e^{-x} = 0$$

$$\therefore 3e^x - ke^{2x} - 1 = 0$$

$$\therefore ke^{2x} - 3e^x + 1 = 0$$

Let $e^x = p$ $\quad kp^2 - 3p + 1 = 0$

For one or more solutions, $\Delta \geq 0$

$$\therefore (3)^2 - 4(k)(1) \geq 0$$

$$9 - 4k \geq 0$$

$$\therefore 9 \geq 4k$$

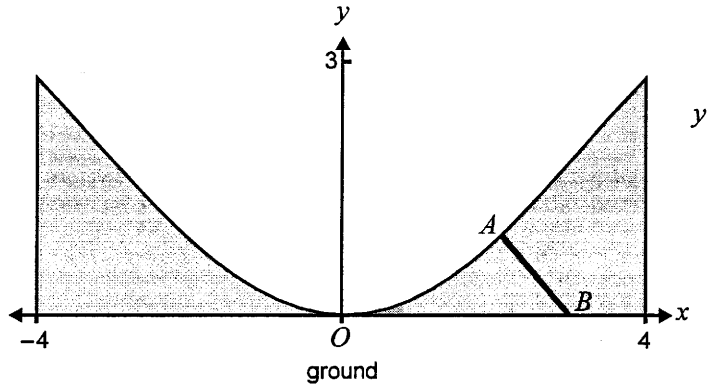
$$\therefore k \leq \frac{9}{4}$$

4 marks

Total 13 marks

Question 2

Andrew is making a skateboard ramp. He draws a cross-section diagram with coordinate axes as shown below.



The curve has the equation $y = 2 - 2\cos\left(\frac{x}{2}\right)$, $-4 \le x \le 4$. All measurements are in metres; the horizontal length of the structure is 8 metres.

a. How many metres above the ground is the highest point of the ramp? Give your answer to two decimal places.

$f(x) = 2 - 2\cos\left(\frac{x}{2}\right), \quad -4 \le x \le 4$

$f(4) = 2 - 2\cos(2)$

$\simeq 2.83$ m

1 mark

b. Show that the gradient of the ramp is always less than or equal to 1.

$f'(x) = 2 \times \frac{1}{2} \sin\left(\frac{x}{2}\right)$

$= \sin\left(\frac{x}{2}\right)$

Since $\left|\sin\left(\frac{x}{2}\right)\right| \le 1$ for all $x \in \mathbb{R}$,

$-1 \le f'(x) \le 1$ for all $x \in [-4, 4)$

2 marks

c. **i.** Write a definite integral which gives the area of the shaded region.

$$\int_{-4}^{4}\left(2-2\cos\left(\frac{x}{2}\right)\right)dx$$

ii. Find the area of the shaded region, correct to two decimal places.

$$\left[2x - 4\sin\left(\frac{x}{2}\right)\right]_{-4}^{4}$$

$$= (8 - 4\sin(2)) - (-8 - 4\sin(-2)) \approx 8.73\ m^2$$

2 + 1 = 3 marks

There is a supporting beam *AB* on the structure as shown. *A* is a point on the curve one metre vertically above the *x*-axis. *B* is a point on the *x*-axis such that *AB* is normal to the curve at *A*.

d. **i.** Find the exact *x*-coordinate of *A*.

$$2 - 2\cos\left(\frac{x}{2}\right) = 1 \qquad -4 \le x \le 4$$

$$-2\cos\left(\frac{x}{2}\right) = -1 \qquad -2 \le \frac{x}{2} \le 2$$

$$\cos\left(\frac{x}{2}\right) = \frac{1}{2}$$

$$\therefore \frac{x}{2} = \frac{\pi}{3}, \text{ so } x = \frac{2\pi}{3} \quad \left(\frac{2\pi}{3}, 1\right)$$

ii. Find the exact value of the gradient of the normal to the curve at *A*.

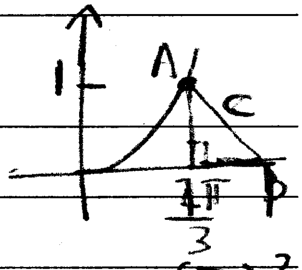
$$f'(x) = \sin\left(\frac{x}{2}\right)$$

$$\therefore f'\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\therefore m_{NORMAL} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

iii. Find the exact length of *AB*.

Find co-ordinates of B: Let $B = (b, 0)$



$A = \left(\frac{2\pi}{3}, 1\right)$

$$\therefore \frac{1-0}{\frac{2\pi}{3}-b} = \frac{-2\sqrt{3}}{3}$$

$$3 = -2\sqrt{3}\left(\frac{2\pi}{3}-b\right)$$

$$-\frac{3}{2\sqrt{3}} = \frac{2\pi}{3} - b$$

$$\therefore b - \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$c^2 = 1^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$c^2 = 1 + \frac{3}{4}$$

$$c^2 = \frac{7}{4}$$

$$\therefore c = \frac{\sqrt{7}}{2} \text{ units.}$$

2 + 2 + 3 = 7 marks

Total 13 marks

TURN OVER

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

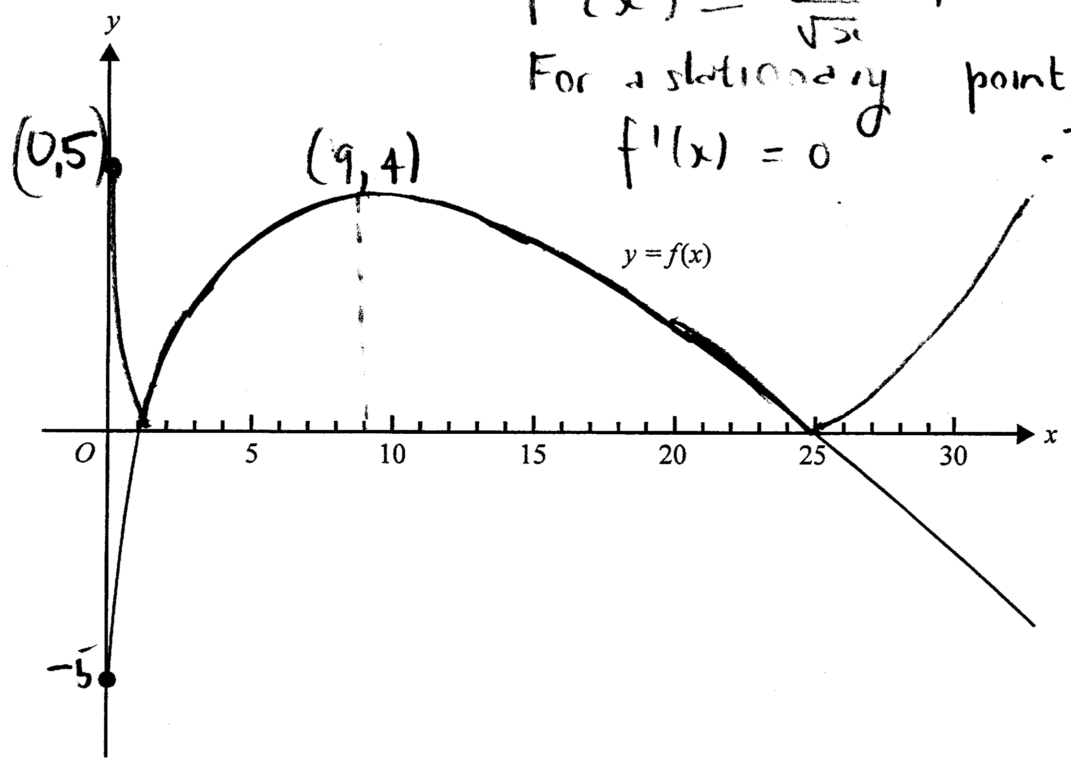
Question 1

Let $f: R^{+} \cup \{0\} \rightarrow R$, $f(x) = 6\sqrt{x} - x - 5$.
The graph of $y = f(x)$ is shown below.

$f'(x) = \frac{3}{\sqrt{x}} - 1$

For a stationary point,

$f'(x) = 0$ $\therefore x = 9$



a. State the interval for which the graph of f is strictly decreasing.

$x \in [9, \infty)$

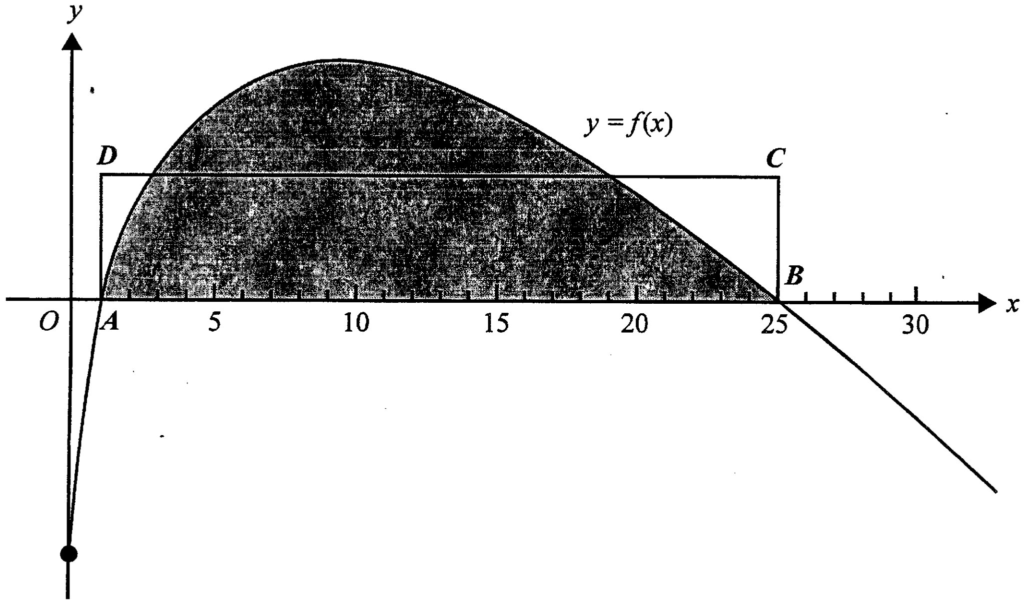
2 marks

b. On the set of axes above, sketch the graph of $y = |f(x)|$.

2 marks

c. Points A and B are the points of intersection of $y = f(x)$ with the x-axis. Point A has coordinates (1, 0) and point B has coordinates (25, 0).

Find the length of AD such that the area of rectangle $ABCD$ is equal to the area of the shaded region.



We are required to find the average value of the function over [1,25]

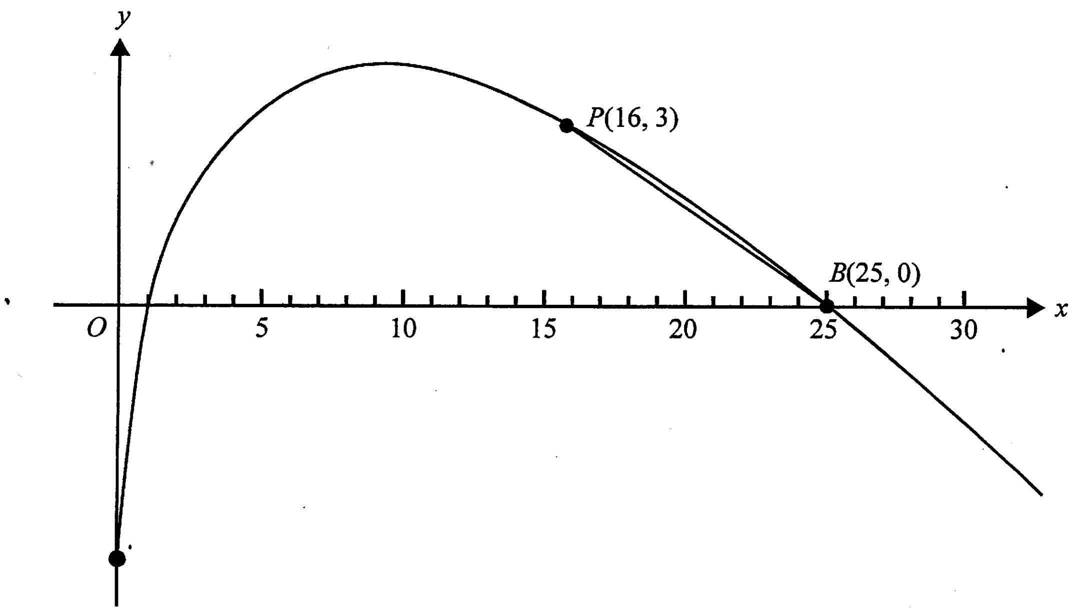
$$\frac{1}{25-1}\int_1^{25}\left(6\sqrt{x}-x-5\right)dx$$

$$=\frac{1}{24}\int_1^{25}\left(6\sqrt{x}-x-5\right)dx$$

$$=\frac{8}{3}$$

2 marks

d. The points $P(16, 3)$ and $B(25, 0)$ are labelled on the diagram.



i. Find m, the gradient of the chord PB. (Exact value to be given.)

$$\frac{3-0}{16-25} = \frac{3}{-9} = -\frac{1}{3}$$

ii. Find $a \in [16, 25]$ such that $f'(a) = m$. (Exact value to be given.)

$$f'(x) = \frac{3}{\sqrt{x}} - 1$$

$$f'(a) = \frac{3}{\sqrt{a}} - 1$$

1 + 2 = 3 marks

$$\frac{3}{\sqrt{a}} - 1 = -\frac{1}{3}$$

$$\frac{3}{\sqrt{a}} = \frac{2}{3}$$

$$9 = 2\sqrt{a}$$

$$a = \frac{81}{4}$$

e. Let $g: R \to R,\ g(x) = x^2$.

i. Find the rule of $f(g(x))$.

$f(g(x)) = 6\sqrt{x^2} - x^2 - 5$
$= 6|x| - x^2 - 5$

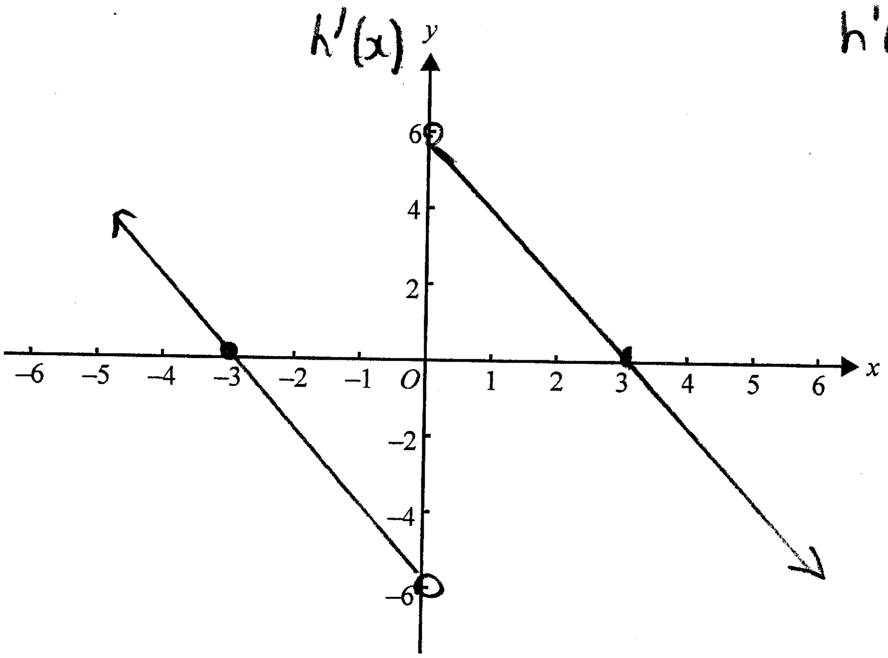
Let $h(x) = f(g(x))$.

ii. Find the derivative of $h(x)$ with respect to x.

$h(x) = 6|x| - x^2 - 5$ where $x \in \mathbb{R}$
$h(x) = 6x - x^2 - 5,\ x \geq 0$
$= -6x - x^2 - 5,\ x < 0$

iii. Sketch the graph of the derivative function $y = h'(x)$ on the axes provided below.

so
$h'(x) = 6 - 2x,\ x > 0$
$= -6 - 2x,\ x < 0$
undefined at $x = 0$



2 + 2 + 3 = 7 marks

Total 16 marks