Question 1

a. Differentiate the function: $f(x) = (x^2 + 6x + 3)e^{2x}$ with respect to x.

Express your final answer in the form $Q(x)e^{2x}$ where $Q(x)$ is a quadratic function.

$$f(x) = (x^2+6x+3)e^{2x}$$

$u = x^2+6x+3$

$u' = 2x+6$

$v = e^{2x}$

$v' = 2e^{2x}$

$$\therefore f'(x) = e^{2x}(2x+6) + 2e^{2x}(x^2+6x+3)$$
$$= e^{2x}(2x+6 + 2x^2+12x+6)$$
$$= e^{2x}(2x^2+14x+12)$$

2 marks

b. Find the exact co-ordinates of any stationary points.

For stationary points, $f'(x)=0$

$$\therefore e^{2x}(2x^2+14x+12)=0$$

$e^{2x} \neq 0$ for all $x \in \mathbb{R}$

$$\therefore 2x^2+14x+12=0$$
$$x^2+7x+6=0$$
$$(x+6)(x+1)=0 \quad \therefore x=-6,-1$$

If $x=-6$, $f(-6) = e^{-12}(36-36+3) = 3e^{-12}$

If $x=-1$, $f(-1) = (1-6+3)e^{-2} = -2e^{-2}$

3 marks

Stationary points: $(-6, 3e^{-12})$, $(-1, -2e^{-2})$

Question 2

Given that: $\int_{\frac{1}{2}}^{p}(2x-1)^3\,dx = 2$, where $p > \frac{1}{2}$, find the value of p.

$$\int_{\frac{1}{2}}^{p}(2x-1)^3\,dx = 2$$

$$\therefore \left[\frac{(2x-1)^4}{4\times 2}\right]_{\frac{1}{2}}^{p} = 2$$

$$\therefore \frac{(2p-1)^4}{8} - 0 = 2$$

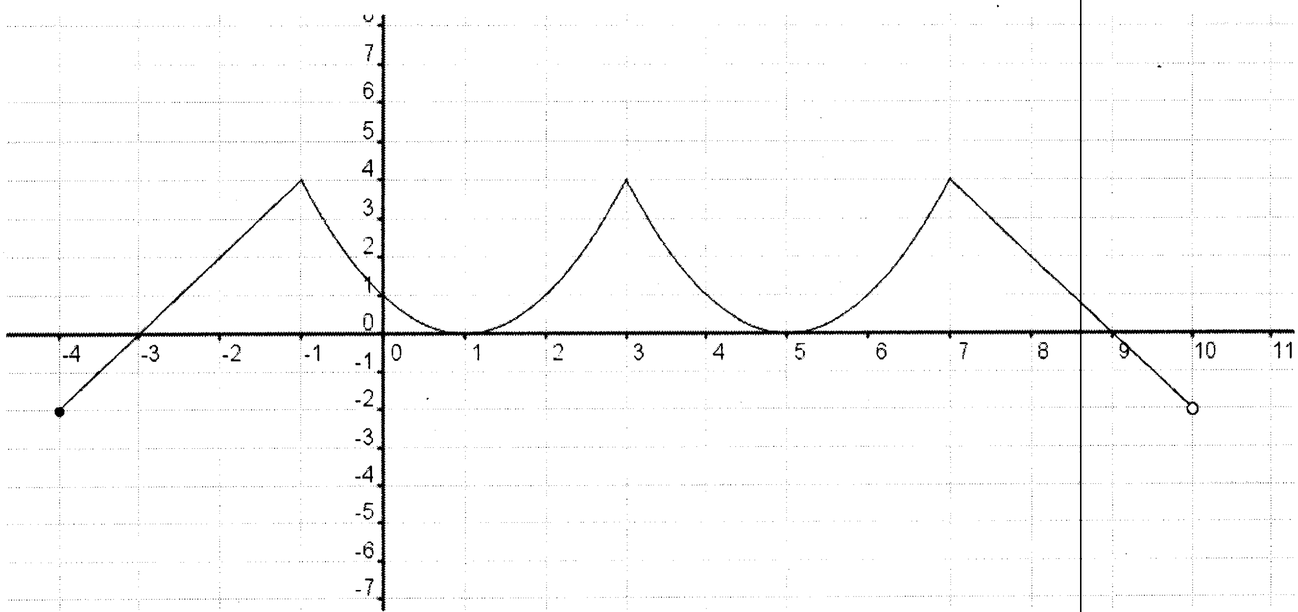
$$\frac{(2p-1)^4}{8} = 2$$
$$(2p-1)^4 = 16$$
$$2p-1 = \pm 2$$
$$2p = 3, -1$$
$$p = \frac{3}{2}, -\frac{1}{2}$$

But $p > \frac{1}{2}$ $\therefore p = \frac{3}{2}$

3 marks

Question 3

The diagram below shows the graph of a hybrid function $f(x)$. The function $f(x)$ consists of two line segments and two quadratic functions joined together.



a. What is the domain of $f(x)$? $[-4, 10)$

1 mark

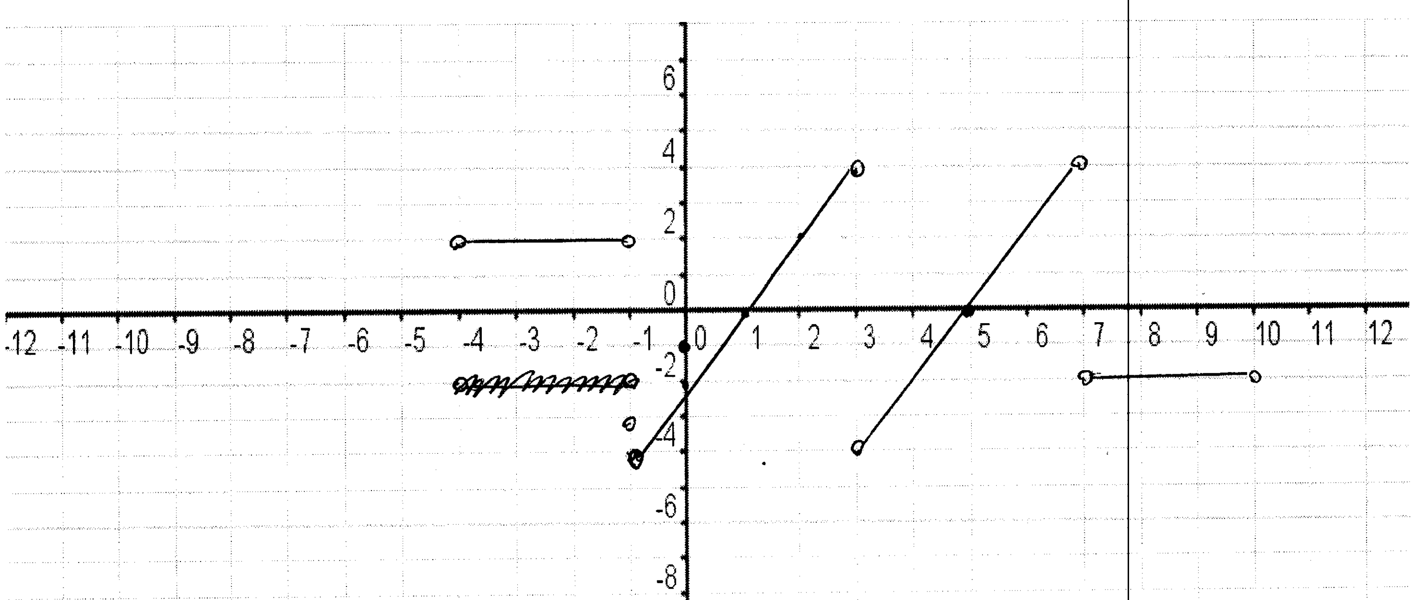
b. State the equations of the two quadratic sections of the function $f(x)$.

$y = (x-1)^2$

$y = (x-5)^2$

2 marks

c. On the set of axes below, sketch the graph of the derivative function $f'(x)$.



$y=(x-1)^2 \therefore \frac{dy}{dx} = 2(x-1)$ $\quad$ $y=(x-5)^2 \therefore \frac{dy}{dx} = 2(x-5)$

3 marks

c. State the domain of the derivative function.

$x \in (-4, 10) \setminus \{-1, 3, 7\}$

1 mark

Question 4

Let $g(x) = \frac{\log_e x}{x}$

a. Find $g'(x)$.

$y = \frac{u}{v}$ $u = \log_e x$, $v = x$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$\therefore \frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \log_e x \times 1}{x^2}$

$= \frac{1 - \log_e x}{x^2}$

2 marks

b. Hence find the exact value of $\int_1^e \frac{\log_e x}{x^2}\, dx$

Find: $\int_1^e \frac{\log_e x}{x^2}\, dx$

We know: $\frac{d}{dx}\left(\frac{\log_e x}{x}\right) = \frac{1 - \log_e x}{x^2}$

$\therefore \frac{d}{dx}\left(\frac{\log_e x}{x}\right) = \frac{1}{x^2} - \frac{\log_e x}{x^2}$

$\therefore \frac{\log_e x}{x} = \int \frac{1}{x^2}\, dx - \int \frac{\log_e x}{x^2}\, dx$

$\therefore \int \frac{\log_e x}{x^2}\, dx = \int \frac{1}{x^2}\, dx - \frac{\log_e x}{x}$

$\therefore \int \frac{\log_e x}{x^2}\, dx = -\frac{1}{x} - \frac{\log_e x}{x}$

$\therefore \int_1^e \frac{\log_e x}{x^2}\, dx = \left[-\frac{1}{x} - \frac{\log_e x}{x}\right]_1^e = \left[-\frac{1}{e} - \frac{\log_e e}{e}\right] - \left[-\frac{1}{1} - \frac{\log_e 1}{1}\right]$

$= 1 - \frac{1}{e} - \frac{1}{e} = 1 - \frac{2}{e}$

3 marks