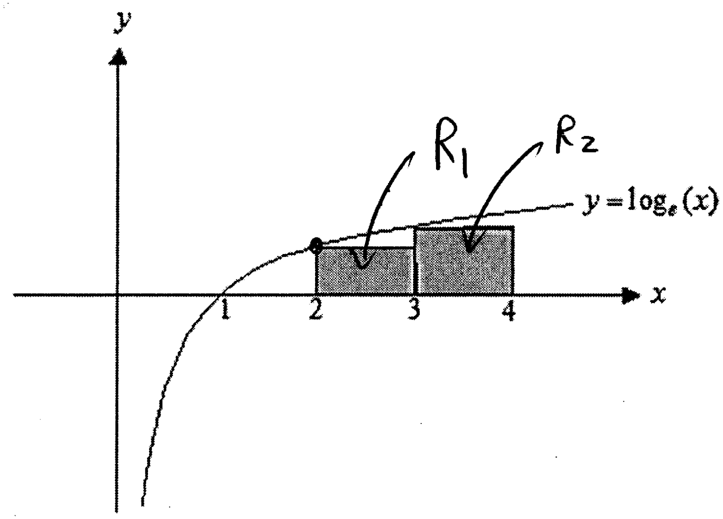
SOLUTIONS.

Question 1



The area under the curve $y = \log_e(x)$ between $x = 1$ and $x = 4$ is approximated by the two shaded rectangles shown above. This approximate area in square units is

A. $\log_e(1.5)$
B. $\log_e(5)$
C. $\log_e(6)$
D. $2\log_e(2)$
E. $3\log_e(2)$

Area of $R_1 = f(2) \times 1$
$= \log_e 2 \times 1$

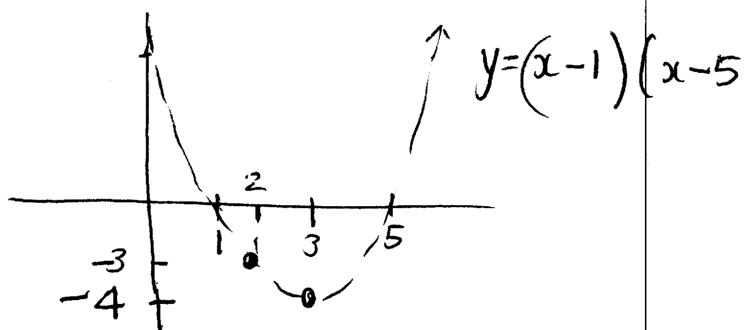
Area of $R_2 = f(3) \times 1$
$= \log_e 3$

$\therefore$ Total $= \log_e 2 + \log_e 3 = \log_e 6$

Question 2

The average rate of change of the function $y = |(x-1)(x-5)|$ between $x = 2$ and $x = 3$ is

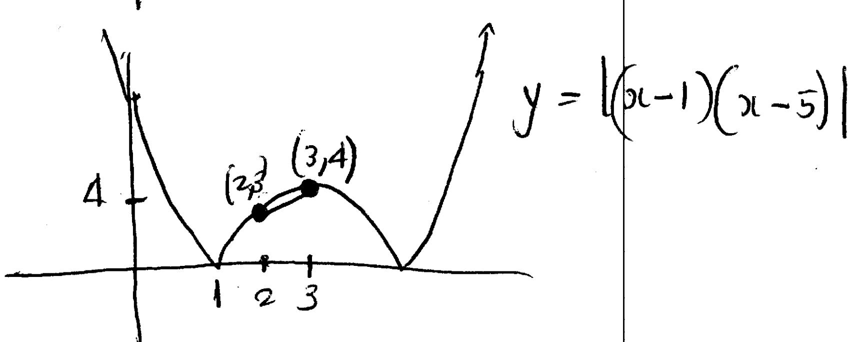
A. -1
B. 0
C. $\frac{1}{6}$
D. 1
E. 7



Av. rate of change

$= \frac{4-3}{3-2}$

$= 1$



Question 3

The average value of the function $y = \sin(2x)$ over the interval $[\frac{\pi}{6}, \frac{\pi}{3}]$ is

A. $\frac{3}{\pi}$

B. $\frac{6}{\pi}$

C. $\frac{3}{2\pi}$

D. $\frac{1}{\pi}$

E. $\frac{1}{2}$

$$\frac{1}{\frac{\pi}{3}-\frac{\pi}{6}} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin(2x)\,dx$$

$$= \frac{6}{\pi} \int_{\pi/6}^{\pi/3} \sin(2x)\,dx$$

$$= \frac{6}{\pi} \times \frac{1}{2} = \frac{3}{\pi}$$

Question 4

The radius of a sphere is increasing at the rate of 4 mm/min . At the instant when the radius of the sphere is 5mm, the rate at which the surface area of the sphere is increasing, in mm^2/min, is :

A. 160π
B. 32π
C. 400π
D. 160
E. 100π

Variables S = surface area
r = radius
t = time

We want: $\frac{dS}{dt}$

$S = 4\pi r^2$

$\therefore \frac{dS}{dr} = 8\pi r$

$$\frac{dS}{dt} = \frac{dS}{dr}\,\frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi r \times 4$$

$$= 32\pi r.$$ When $r = 5$, $\frac{dS}{dt} = 5 \times 32\pi = 160\pi$

Question 5

An industrial chemical storage tank has been damaged by an earth tremor, and toxic chemicals are now leaking from it. Engineers at the site have estimated that the rate at which the chemicals are leaking from the tank at any time t is equal to $6\sqrt{t}$ litres per minute. If there are initially 19 652 litres of chemicals in the tank, the time required for all the chemicals to escape is closest to:

A. 57 minutes
B. 70 minutes
C. 88 minutes
D. 289 minutes
E. 728 minutes

$\frac{dV}{dt} = -6\sqrt{t}$ V = volume in tank

$\therefore \frac{dV}{dt} = -6t^{\frac{1}{2}}$

$V = \int -6\sqrt{t}\,dt$

$= \frac{-6t^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= -4t^{\frac{3}{2}} + c$

When $t = 0$, $V = 19652$

$\therefore V = 19652 - 4t^{3/2}$

When $V = 0$

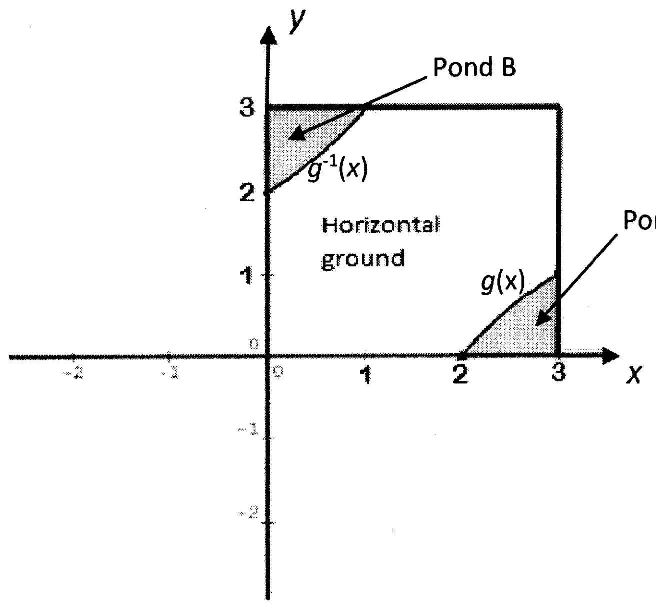
$19652 - 4t^{3/2} = 0$

$\therefore t^{3/2} = \frac{19652}{4}$

$\therefore t = 289$

Question 6

An ornamental garden is designed in a 3m by 3m square. It has two ponds A and B, separated by horizontal ground, as represented in the diagram below:



The curved border of Pond A is modelled by the function: $g(x) = \log_2(x-1)$, and the curved border of Pond B is modelled by the inverse function of $g(x)$, $g^{-1}(x)$. The designers of the garden have decided to cover the horizontal ground between the ponds with artifical lawn. If the thickness of the lawn is 0.15 metres, then the volume of the artificial lawn required, in cubic metres correct to two decimal places is equal to:

A. 8.44

B. 7.89

C. 1.52

D. 1.27

E. 1.18

~~$g(x) = \log_2(x-1)$~~

~~$\therefore y = \log_2(x-1)$~~

To find g^{-1}:

~~$x = \log_2(y-1)$~~

~~$\therefore y = 2^x$~~

Area of Pond A = Area of Pond B (by symmetry)

Area of Pond A

$$= \int_2^3 \log_2(x-1)\,dx$$

$$= 0.557305$$

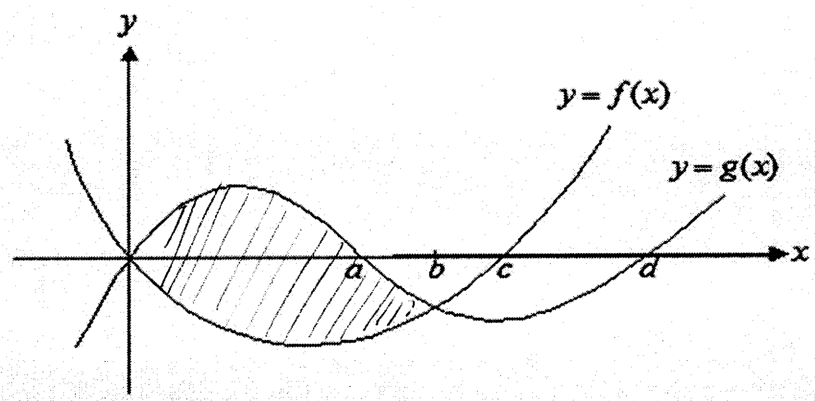
$\therefore$ Area to be covered by lawn

$$= 9 - 2 \times 0.557305 = 7.88539 \text{ m}^2$$

$\therefore$ Volume $= 7.88539 \times 0.15 = 1.18 \text{ m}^3$

Question 7

The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



The area enclosed by these two graphs is given by

A. $\int_0^b (g(x) - f(x))dx$

B. $\int_0^b (g(x) + f(x))dx$

C. $\int_0^a (g(x))dx - \int_0^b f(x)dx$

D. $\int_0^a (g(x))dx - \int_0^c f(x)dx$

E. $\int_0^d (g(x))dx - \int_0^c f(x)dx$

Area enclosed

$= \int_0^b f_{upper} - \cancel{g} f_{lower} \, dx$

$= \int_0^b (g(x) - f(x))dx$

Question 8

The function g is continuous and differentiable for $x \in R$. The function g satisfies the following conditions.

- $f'(x) = 0$ where $x = -1,\ 0$ and 4
- $f'(x) > 0$ where $x < -1$ and $0 < x < 4$
- $f'(x) < 0$ where $-1 < x < 0$ and $x > 4$

It is true to say that the graph of $y = g(x)$ has

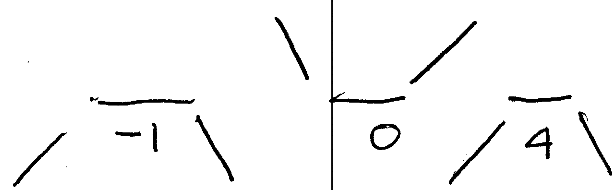
A. a stationary point of inflection where $x = 0$

B. a point of inflection where $x = 0$

C. a local maximum where $x = 0$

D. a local minimum where $x = 0$

E. two local minimum



Question 9

The function g is a smooth, continuous function for $x \in R$ and $g(x) \geq 0$ for $x \in [0, k]$.

Given that $\int_0^k g(x)dx = 2k$, then $4\int_0^{3k}\left(g\left(\frac{x}{3}\right)-1\right)dx$ is equal to

A. 0

B. $12k$

C. $20k$

D. $12k - 4$

E. $24k - 4$

$4\int_0^{3k} g\left(\frac{x}{3}\right)-1\,dx$

$= 4\int_0^{3k} g\left(\frac{x}{3}\right)dx - \int_0^{3k} 4\,dx$

$= 4\times 6k - [4x]_0^{3k}$

$= 24k - (12k - 0)$

$= 12k$

The function $g\left(\frac{x}{3}\right)$ is dilated by a factor of 3 parallel to the x-axis.

$\therefore \int_0^{3k} g\left(\frac{x}{3}\right)dx = 3\times 2k$

$= 6k$

Question 10

Using the approximation formula: $f(x+h) \approx f(x) + hf'(x)$ with $f(x) = x^{\frac{3}{2}}$, an approximate value for $15.9^{\frac{3}{2}}$ would be:

A. $f(16) + \frac{1}{10} \times \frac{3}{2} \times \sqrt{16}$

B. $f(64) - \frac{1}{10} \times \frac{3}{2} \times \sqrt{64}$

C. $f(16) - \frac{1}{10} \times \frac{3}{2} \times \sqrt{16}$

D. $f(64) + \frac{1}{10} \times \frac{3}{2} \times \sqrt{64}$

E. $f(64) - f(15.9)$

Let $f(x) = x^{\frac{3}{2}}$

$f(15.9) \approx f(16) - 0.1 \times f'(16)$

$f'(x) = \frac{3}{2}x^{1/2}$

$\therefore f(15.9) \approx f(16) - 0.1 \times \frac{3}{2} \times \sqrt{16}$

Question 11

If $h(x) = x^2 + |x|$, then the derivative function of $h(x)$, $h'(x)$ would be:

A. $\begin{cases} 2x+1, & x>0 \\ 2x-1 & x<0 \end{cases}$

B. $\begin{cases} 2x+1, & x>0 \\ -2x-1 & x<0 \end{cases}$

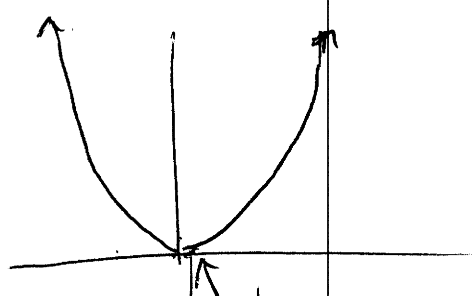
C. $\begin{cases} 2x+1, & x\geq 0 \\ 2x-1 & x<0 \end{cases}$

D. $\begin{cases} 2x+1, & x\geq 0 \\ -2x-1 & x<0 \end{cases}$

E. $\begin{cases} 2x-1, & x>0 \\ 2x+1 & x<0 \end{cases}$

$h(x) = x^2 + x, \quad x \geq 0$

$= x^2 - x, \quad x < 0$



sharp point at $x=0$

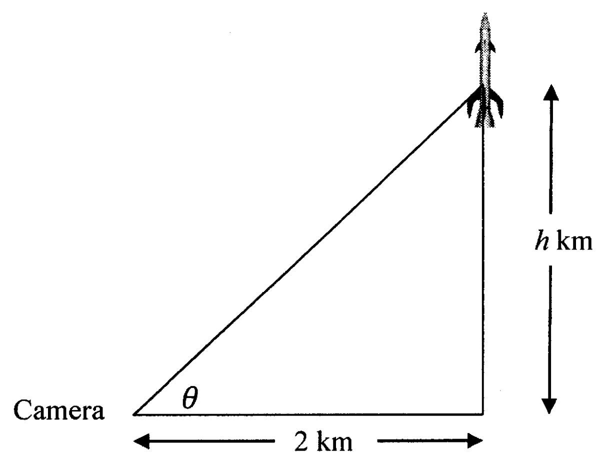
$h'(x) = 2x+1 \quad x>0$

$= 2x-1, \quad x<0$

$h(x)$ is not differentiable at $x=0$.

Question 12

A rocket rises vertically from its launch pad, and as it does so a camera is directed towards it. The camera is located at a distance of 2 km from the launch pad on horizontal ground. The height of the rocket at any instant is h km and is increasing at the rate of 0.1 km/sec. As the rocket rises, the camera rotates through an angle of θ relative to the horizontal.



Variables: θ, h, t

$\frac{dh}{dt} = 0.1$

At the instant when the angle of rotation θ is equal to $\frac{\pi}{6}$ radians, the angle is increasing at the rate of:

Find : $\frac{d\theta}{dt}$

A. $\frac{1}{80}$ radians/sec

B. $\frac{3}{80}$ radians/sec

C. $\frac{1}{20}$ radians/sec

D. $\frac{\pi}{12}$ radians/sec

E. 0.8 radians/sec

$$\frac{d\theta}{dt} = \frac{d\theta}{dh}\,\frac{dh}{dt}$$

$$\frac{d\theta}{dt} = \frac{d\theta}{dh} \times 0.1$$

$$\tan\theta = \frac{h}{2}$$

$$\therefore h = 2\tan\theta$$

$$\therefore \frac{dh}{d\theta} = \frac{2\sec^2\theta}{\cancel{\text{ }}} = \frac{2}{\cos^2\theta}$$

$$\therefore \frac{d\theta}{dh} = \frac{\cos^2\theta}{2}$$

$$\therefore \frac{d\theta}{dt} = \frac{\cos^2\theta}{2} \times \frac{1}{10} = \frac{\cos^2\theta}{20}$$

When $\theta = \frac{\pi}{6}$,

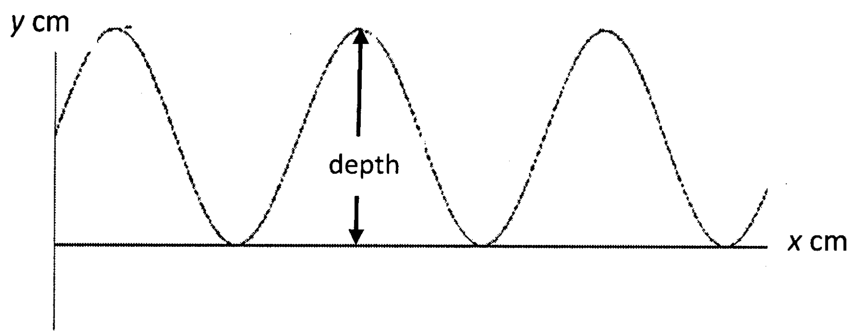
$$\frac{d\theta}{dt} = \frac{\left(\cos\left(\frac{\pi}{6}\right)\right)^2}{20}$$

$$= \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{20}$$

$$= \frac{3}{80} \text{ radians/sec}$$

Question 1

The corrugations in a particular type of roofing form a curve which can be modelled by the function $y = 2\sin(3x) + 2$ where x and y are measured in centimetres. The graph of this function is shown below:



a. What is the period of this function?

Period = $\frac{2\pi}{3}$

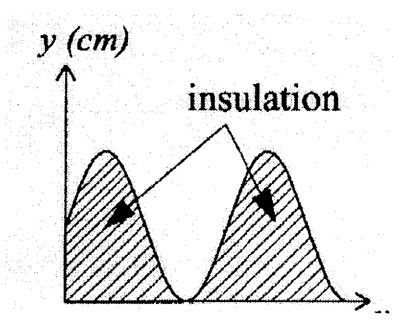
1 mark

b. The depth of a corrugation is the distance from the highest point to the lowest point. What is the depth of corrugations?

4 cm

1 mark

A builder decides to put insulation under the roof in one section. The insulation is to completely fill in the shape of the corrugations. A cross-section of the insulated roof looks like the following diagram:



c. Write down a definite integral to represent the area of cross section shaded if the insulation is used for 10 cycles of the curve.

$10\int_0^{2\pi/3} \left(2\sin(3x) + 2\right)\,dx$

1 mark

d. Use calculus to evaluate this area exactly.

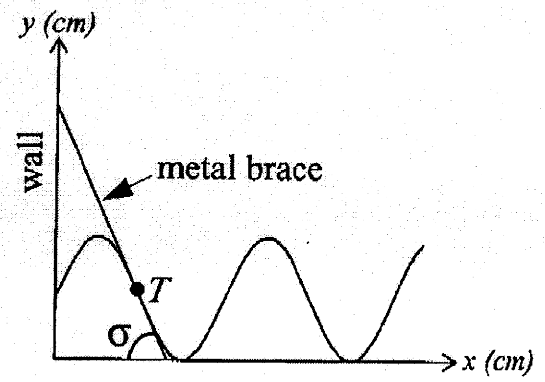
$$10\int_0^{2\pi/3}(2\sin 3x + 2)\,dx = 10\left[-\frac{2}{3}\cos(3x)+2x\right]_0^{2\pi/3}$$

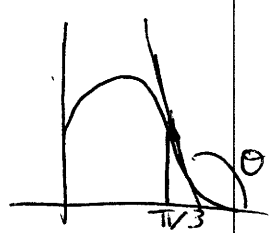
$$= 10\left[-\frac{2}{3}\cos(2\pi)+\frac{4\pi}{3}\right] - 10\left[-\frac{2}{3}\cos 0 + 0\right]$$

$$= -\frac{20}{3}+\frac{40\pi}{3}+\frac{20}{3} = \frac{40\pi}{3} \text{ sq. units}$$

2 marks

Where the corrugated roof touches an existing wall, a metal brace is constructed which is tangent to the curve at the point T where $x = \frac{\pi}{3}$.





$\tan(\theta) = f'\left(\frac{\pi}{3}\right)$

e. Find the angle σ that the metal brace makes with the horizontal. Give your answer to the nearest degree.

$f'(x) = 6\cos(3x)$

$\therefore f'\left(\frac{\pi}{3}\right) = 6\cos(\pi) = -6.$

$\tan(\theta) = -6 \quad \therefore \theta = 99.46^\circ$

$\therefore \sigma = 180^\circ - 99.46^\circ \simeq 81^\circ$

2 marks

f. Find the equation of the line created by the brace.

When $x = \frac{\pi}{3}$, $f\left(\frac{\pi}{3}\right) = 2\sin(\pi) + 2 = 2$

$\therefore$ Tangent passes through $\left(\frac{\pi}{3}, 2\right)$ and has a gradient of -6.

$\therefore y - 2 = -6\left(x - \frac{\pi}{3}\right)$

$y = -6x + 2\pi + 2$

2 marks

g. How far up the wall does the brace reach? (Give your answer in centimetres correct to two decimal places.)

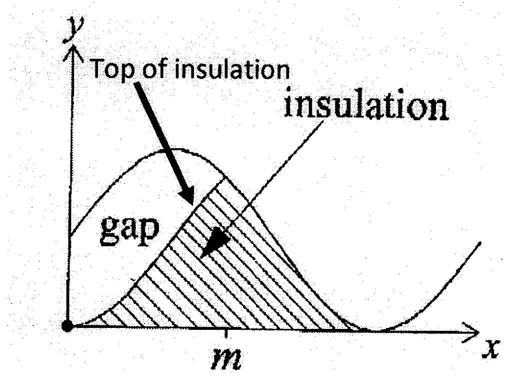
y-intercept; let $x = 0$

$y = 2\pi + 2$

$\simeq 8.28$ cm

1 mark

Suppose that the insulation of the roof compresses down over time and there is a gap between the insulation and the top of the roof, as shown on the diagram below:



The shape of the top of the insulation is given by: $y = -2\cos(3x) + 2$, where $0 \leq x \leq m$, and $m < \frac{\pi}{2}$.

h.i. Show that to find the exact value of m, the equation to be solved is of the form:

$$\tan(3x) = k$$

and state the value of k.

$2\sin(3x) + 2 = -2\cos(3x) + 2$

$\therefore 2\sin(3x) = -2\cos(3x)$

$\therefore \tan(3x) = -1$

$(k = -1)$

$k = -1$

2 marks

ii. Hence, find the exact value of m.

$\tan(3x) = -1 \quad \therefore 3x = \frac{3\pi}{4}$

$\therefore x = \frac{\pi}{4}$

1 mark

$\left(0 < x < \frac{\pi}{2}\right)$

i. **i.** Write down the definite integral that would represent the cross-sectional area fo the gap between the roof and the insulation.

$$\int_0^{\pi/4} 2\sin 3x + 2 - (-2\cos 3x + 2) \; dx$$

1 mark

ii. Use calculus to evaluate this area exactly.

$$\int_0^{\pi/4} (2\sin 3x + 2\cos 3x)\, dx$$

$$= \left[\frac{-2}{3}\cos 3x + \frac{2}{3}\sin 3x \right]_0^{\pi/4}$$

$$= \left[\frac{-2}{3}\cos\left(\frac{3\pi}{4}\right) + \frac{2}{3}\sin\left(\frac{3\pi}{4}\right) \right] - \left[\frac{-2}{3}\cos 0 + \frac{2}{3}\sin 0 \right]$$

2 marks

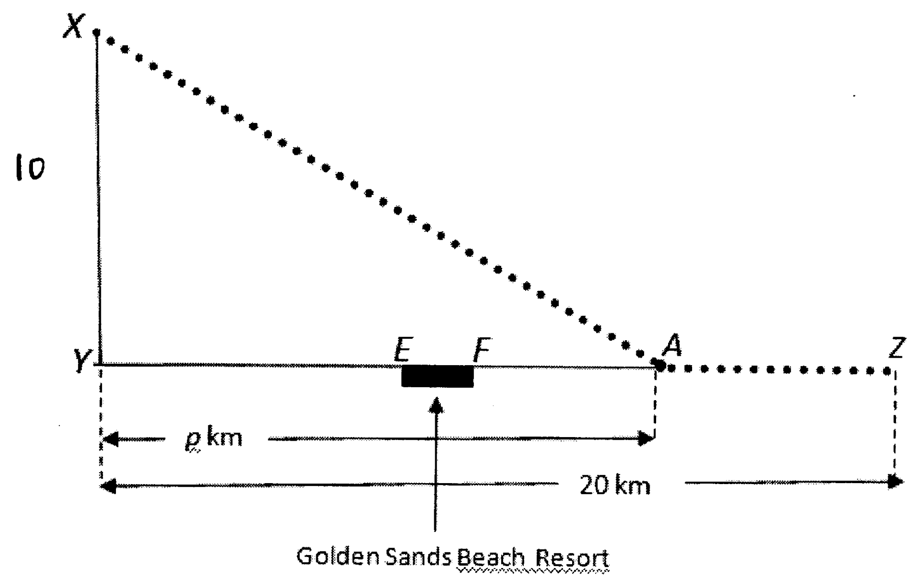
TOTAL 16 marks

$$= \frac{-2}{3} \times -\frac{\sqrt{2}}{2} + \frac{2}{3} \times \frac{\sqrt{2}}{2} + \frac{2}{3}$$

$$= \frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{3} + \frac{2}{3}$$

$$= \frac{2\sqrt{2}}{3} + \frac{2}{3} \quad \text{sq. units}$$

Question 2



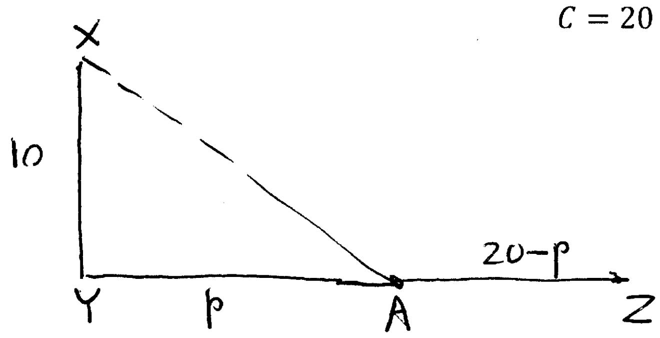
An island is located at X, 10 km from the nearest point Y on a straight beach. Electric power is to be provided by laying a cable between X and a power generation plant located at Z, 20 km along the beach from Y.

The cable contractor decides that the cable will go along the sea bottom from X to A, a point on the beach p kilometres from Y ($p \geq 0$). It will run along the beach to Z.

The cost of laying the cable is \$10000 per kilometre along the beach and $w \times$ \$10000 per kilometre along the sea bottom, where $w > 1$.

a. Let the total cost of laying the cable be $C \times$ \$10000. Show that:

$$C = 20 - p + w\sqrt{(100 + p^2)}$$



Cost = $w\,\overline{XA} + (20-p)\times 1$ (in tens of thousands)

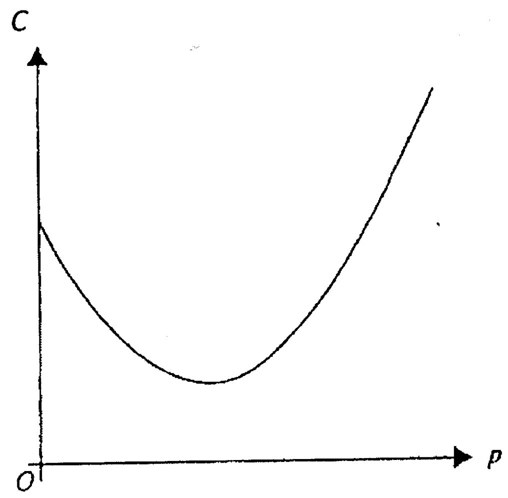
2 marks

~~XA~~ By Pythagorus, $\overline{XA}^2 = p^2 + 10^2$

$XA = \sqrt{100+p^2}$

$\therefore C = w\sqrt{100+p^2} + 20 - p$, where $0 \leq p \leq 20$.

The general shape of the graph of the function representing the cost of laying the cable is shown below. It shows that the function has a single local minimum.



b. Use calculus to show that the cost of laying the cable is a minimum when

$$p^2 = \frac{100}{w^2 - 1}$$

$$C = w\sqrt{100+p^2} + 20 - p$$

$$\frac{dc}{dp} = 2p \times \frac{1}{2} \times \frac{w}{\sqrt{100+p^2}} - 1$$

$$= \frac{pw}{\sqrt{100+p^2}} - 1$$

For a minimum stationary point, $\frac{dc}{dp} = 0$

$$\therefore \frac{pw}{\sqrt{100+p^2}} - 1 = 0$$

$$pw = \sqrt{100+p^2}$$

$$p^2w^2 = 100+p^2$$

$$\therefore p^2(w^2-1) = 100$$

$$\therefore p^2 = \frac{100}{w^2-1}$$

4 marks

The Golden Sands Beach Resort is an expensive holiday destination located on the beach front between E and F as shown in the diagram. E and F are 9 and 10 km from Y respectively.

c. If $w > \sqrt{2}$, show that if the cost of laying is to be a minimum the cable will pass along the beach in front of a part or all of the resort.

The value of p for which a minimum cost occurs will lie to the left of the resort if: $\sqrt{\frac{100}{w^2-1}} < 10$

2 marks

$$\therefore \frac{10}{\sqrt{w^2-1}} < 10$$

$$\therefore \frac{1}{\sqrt{w^2-1}} < 1 \qquad \therefore 1 < \sqrt{w^2-1} \qquad \therefore w^2-1 > 1$$

so $w > \sqrt{2}$

d. **i.** If $w = \sqrt{5}$, find the position of A for which the total cost of laying the cable is a minimum.

If $w = \sqrt{5}$, $p^2 = \dfrac{100}{(\sqrt{5})^2 - 1} = \dfrac{100}{4} = 25$

$\therefore p = \sqrt{25} = 5$

$\therefore$ A is 5 km from Y.

1 mark

ii. The local council has decided to impose a penalty of $20000 on the contractor if the cable passes along any part of the beach in front of the Golden Sands Beach Resort.

Given this penalty, with $w = \sqrt{5}$, determine whether or not it will be cheaper for the contractor to lay the cable so that it does not pass in front of the resort.

If contractor chooses to lay the cable with A at 5km from Y, then total cost $= C(5) + 2$ (in tens of thousands)

$*$ where $C(p) = 20 - p + \sqrt{5}\sqrt{100 + p^2}$

$\therefore C(5) = 20 - 5 + \sqrt{5} \times \sqrt{125} = 15 + \sqrt{625} = 40$

$\therefore$ Total cost $= 40 + 2 =$ ~~\$42,000~~ \$420,000

If instead he avoids running it past the resort, the minimum cost will be if he makes A 10km from Y (just at end of resort).

2 marks

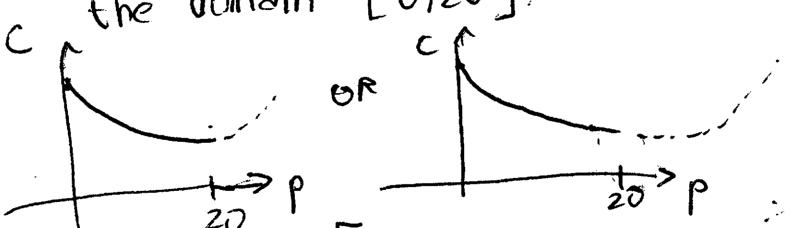
$C(10) = 20 - 10 + \sqrt{5}\sqrt{100 + 10^2} = 10 + \sqrt{1000} = 10 + 31.63$

$= 41.63$. $\therefore$ Cheaper to avoid the penalty.

e. Find the range of values of w for which the cost of laying the cable would be a minimum if the cable were laid in a straight line from X to Z.

(e) It will be cheaper to run from directly X to Z if the graph of C versus p has its t/p either at $p = 20$ or outside the domain $[0, 20]$:

2 marks



TOTAL 13 marks

$\therefore \sqrt{\dfrac{100}{w^2 - 1}} > 20$

$\therefore \dfrac{100}{w^2 - 1} > 400$ $\therefore 100 > 400w^2 - 400$

$\therefore 500 > 400w^2$

$w^2 < \dfrac{5}{4}$ $\therefore w < \dfrac{\sqrt{5}}{2}$

If $1 < w < \dfrac{\sqrt{5}}{2}$ it is cheaper to run in a direct route