SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1
The function with rule $f(x) = -3\tan(2\pi x)$ has period

A. $\frac{2}{\pi}$

B. 2

C. $\frac{1}{2}$

D. $\frac{1}{4}$

E. 2π

$$\text{Period} = \frac{\pi}{2\pi} = \frac{1}{2}$$

Question 2
The midpoint of the line segment that joins $(1, -5)$ to $(d, 2)$ is

A. $\left(\frac{d+1}{2}, -\frac{3}{2}\right)$

B. $\left(\frac{1-d}{2}, -\frac{7}{2}\right)$

C. $\left(\frac{d-4}{2}, 0\right)$

D. $\left(0, \frac{1-d}{3}\right)$

E. $\left(\frac{5+d}{2}, 2\right)$

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{1+d}{2}, \frac{-5+2}{2}\right) = \left(\frac{1+d}{2}, \frac{-3}{2}\right)$$

Question 3
If $x + a$ is a factor of $7x^3 + 9x^2 - 5ax$, where $a \in R\backslash\{0\}$, then the value of a is

A. −4
B. −2
C. −1
D. 1
E. 2

$$P(-a) = 7(-a)^3 + 9(-a)^2 - 5a(-a) = -7a^3 + 9a^2 + 5a^2 = 0$$

$$\therefore -7a^3 + 14a^2 = 0$$

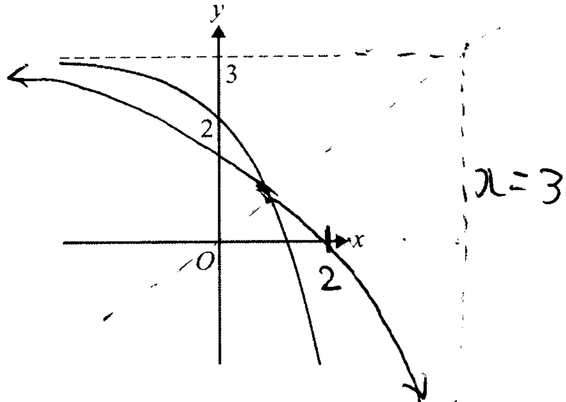
$$-7a^2(a-2) = 0$$

$$a = 2 \text{ (since } a \neq 0)$$

SECTION 1 – continued

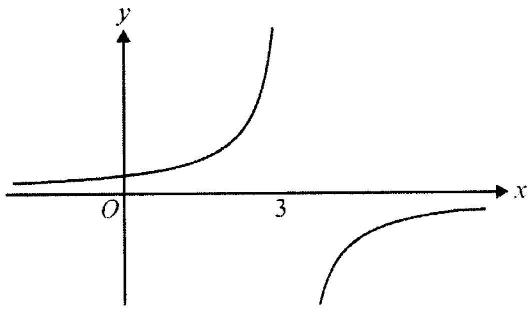
Question 4

Part of the graph of $y = f(x)$, where $f: R \to R$, $f(x) = 3 - e^x$, is shown below.

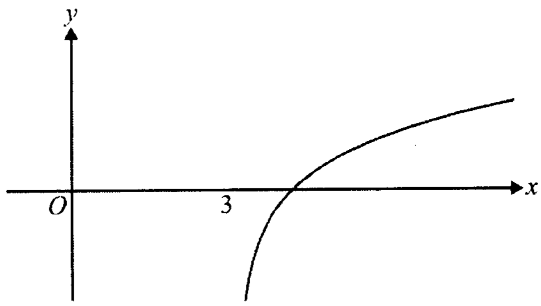


Which one of the following could be the graph of $y = f^{-1}(x)$, where f^{-1} is the inverse of f?

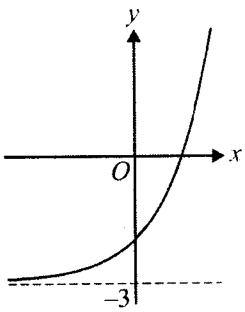
A.



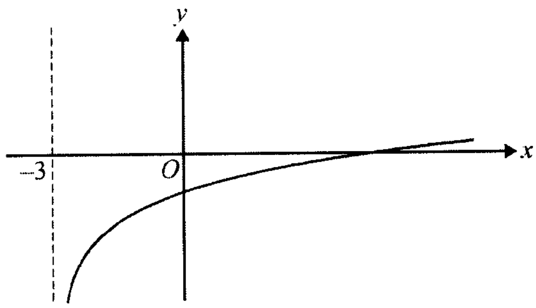
B.



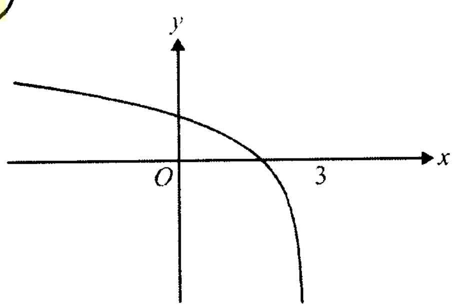
C.



D.



E.



Question 5

If $f: (-\infty, 1) \to R, f(x) = 2\log_e(1 - x)$ and $g: [-1, \infty) \to R, g(x) = 3\sqrt{x+1}$, then the maximal domain of the function $f + g$ is

A. $[-1, 1)$
B. $(1, \infty)$
C. $(-1, 1]$
D. $(-\infty, -1]$
E. R

$\text{dom}(f) = (-\infty, 1) \quad \text{dom}(g) = [-1, \infty)$

$\text{dom}(f) \cap \text{dom}(g) = [-1, 1)$

Question 6

For the function $f(x) = \sin(2\pi x) + 2x$, the average rate of change for $f(x)$ with respect to x over the interval $\left[\frac{1}{4}, 5\right]$ is

A. 0
B. $\frac{34}{19}$
C. $\frac{7}{2}$
D. $\frac{2\pi + 10}{4}$
E. $\frac{23}{4}$

$$\frac{f(5) - f(\frac{1}{4})}{5 - \frac{1}{4}} = \frac{\sin(10\pi) + 10 - \left(\sin(\frac{\pi}{2}) + \frac{1}{2}\right)}{\frac{19}{4}}$$

$$= \frac{10 - \frac{3}{2}}{\frac{19}{4}} = \frac{\frac{17}{2}}{\frac{19}{4}}$$

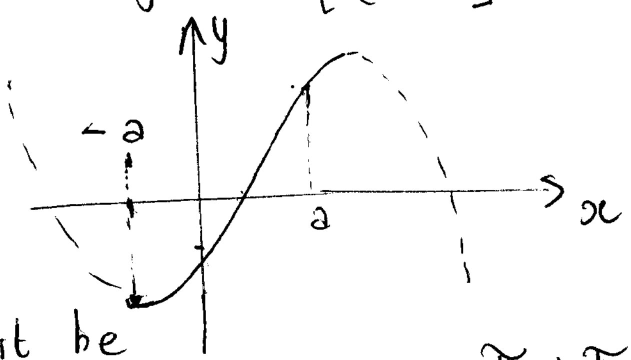
$$= \frac{34}{19}$$

Question 7

The function $g: [-a, a] \to R, g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ has an inverse function.
The maximum possible value of a is

A. $\frac{\pi}{12}$
B. 1
C. $\frac{\pi}{6}$
D. $\frac{\pi}{4}$
E. $\frac{\pi}{2}$

Graph $y = \sin\left[2\left(x - \frac{\pi}{6}\right)\right]$ on CAS



For an inverse, f must be one to one. To be one-one, the value of $-a$ must be no further to the left than the first minimum t/p.

$$\therefore -a = -\frac{\pi}{4} + \frac{\pi}{6}$$

$$= -\frac{\pi}{12}$$

$$\therefore a = \frac{\pi}{12}$$

Question 8

This question involves Markov chains and is no longer on the syllabus

Question 9

Harry is a soccer player who practises penalty kicks many times each day.

Each time Harry takes a penalty kick, the probability that he scores a goal is 0.7, independent of any other penalty kick.

One day Harry took 20 penalty kicks.

Given that he scored at least 12 goals, the probability that Harry scored exactly 15 goals is closest to

A. 0.1789

B. 0.8867

C. 0.8

D. 0.6396

E. 0.2017

X = no of goals

$X \stackrel{d}{=} Bi(n=20, p=0.7)$

$$\Pr(X=15 \mid X \geq 12) = \frac{\Pr(X=15 \cap X \geq 12)}{\Pr(X \geq 12)} = \frac{\Pr(X=15)}{\Pr(X \geq 12)} = 0.2017$$

Question 10

For events A and B, $\Pr(A \cap B) = p$, $\Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$.

If A and B are independent, then the value of p is

A. 0

B. $\frac{1}{4}$

C. $\frac{3}{8}$

D. $\frac{1}{2}$

E. $\frac{3}{5}$

	B	B'
A	p	$\frac{3p}{5}$
A'	$p-\frac{1}{8}$	

Since A, B are independent

$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

$\Pr(B) = p + p - \frac{1}{8} = 2p - \frac{1}{8}$

$\Pr(A) = p + \frac{3p}{5} = \frac{8p}{5}$

$\therefore p = \left(2p - \frac{1}{8}\right) \cdot \frac{8p}{5}$

$\therefore 1 = \frac{8\left(2p - \frac{1}{8}\right)}{5}$

$5 = 16p - 1$

$p = \frac{3}{8}$

Question 11

If the tangent to the graph of $y = e^{ax}$, $a \neq 0$, at $x = c$ passes through the origin, then c is equal to

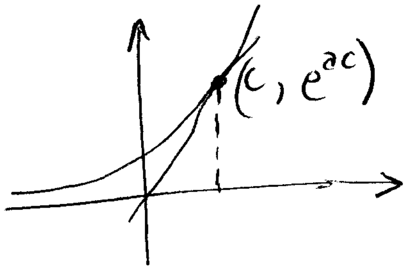
A. 0

B. $\frac{1}{a}$

C. 1

D. a

E. $-\frac{1}{a}$



$y = e^{ax}$

$\frac{dy}{dx} = ae^{ax}$

$\left.\frac{dy}{dx}\right|_{x=c} = ae^{ac}$

$\therefore ae^{ac} = \frac{e^{ac} - 0}{c - 0}$

$\therefore cae^{ac} = e^{ac}$

$\therefore ac = 1$

$\therefore c = \frac{1}{a}$

Question 12

This question involves related rates and is no longer on the syllabus.

Question 13

If the equation $f(2x) - 2f(x) = 0$ is true for all real values of x, then the rule for f could be

A. $\frac{x^2}{2}$

B. $\sqrt{2x}$

C. $2x$

D. $\log_e\left(\frac{|x|}{2}\right)$

E. $x - 2$

Test A: $f(2x) - 2f(x)$

$= \frac{(2x)^2}{2} - 2\left(\frac{x^2}{2}\right)$

$= 2x^2 - x^2 \neq 0$

Test B: $f(2x) - 2f(x)$

$= \sqrt{2 \times 2x} - 2\sqrt{2x}$

$= 2\sqrt{x} - 2\sqrt{2x} \neq 0$

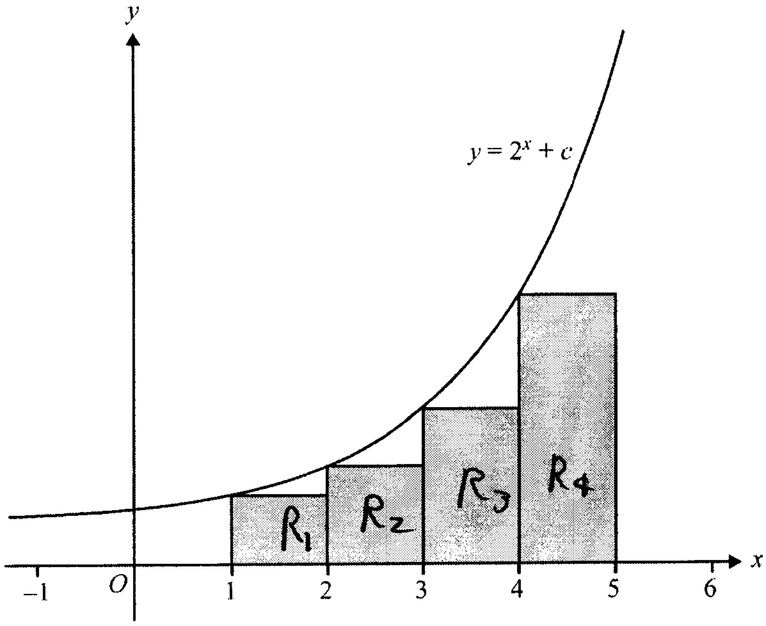
Test C: $f(2x) - 2f(x)$

$= 2(2x) - 2 \times 2x$

$= 0$

Question 14

Consider the graph of $y = 2^x + c$, where c is a real number. The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the x-axis and the lines $x = 1$ and $x = 5$.



If the total area of the shaded rectangles is 44, then the value of c is

A. 14

B. −4

C. $\frac{14}{5}$

D. $\frac{7}{2}$

E. $-\frac{16}{5}$

Area of $R_1 = f(1) \times 1 = f(1)$

$R_2 = f(2)$

$R_3 = f(3)$

$R_4 = f(4)$

$\therefore 44 = (2^1 + c) + (2^2 + c) + (2^3 + c) + (2^4 + c)$

$\therefore 44 = 2 + 4 + 8 + 16 + 4c$

$\therefore 44 = 30 + 4c$

$14 = 4c$

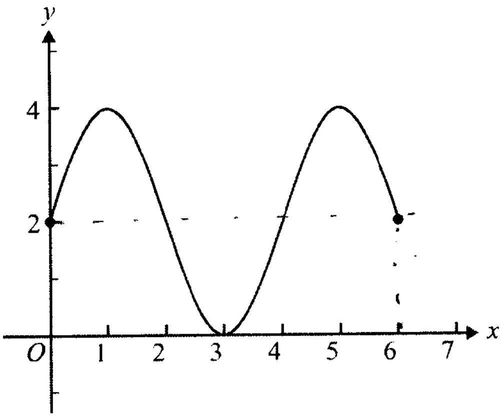
$c = \frac{7}{2}$

Question 15

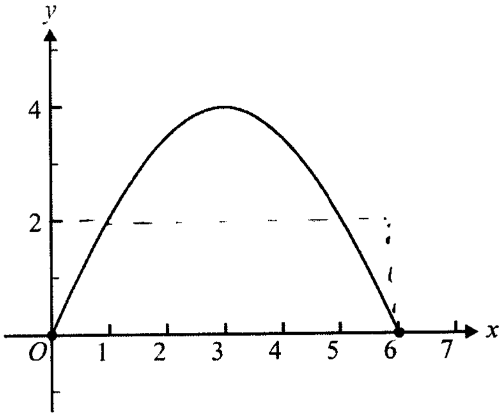
Let h be a function with an average value of 2 over the interval $[0, 6]$.

The graph of h over this interval could be

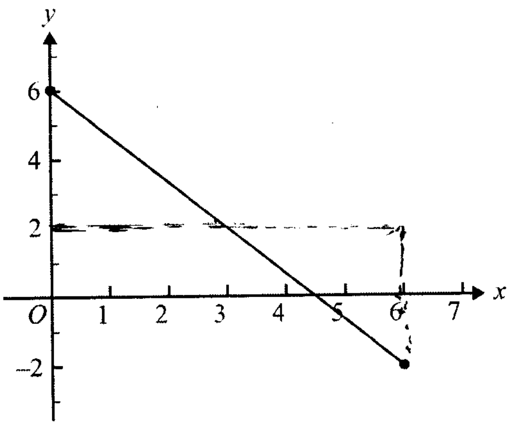
A.



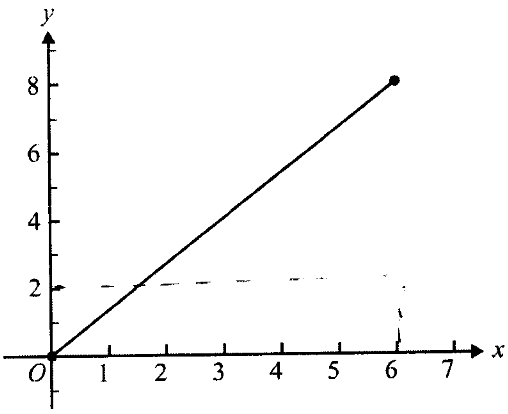
B.



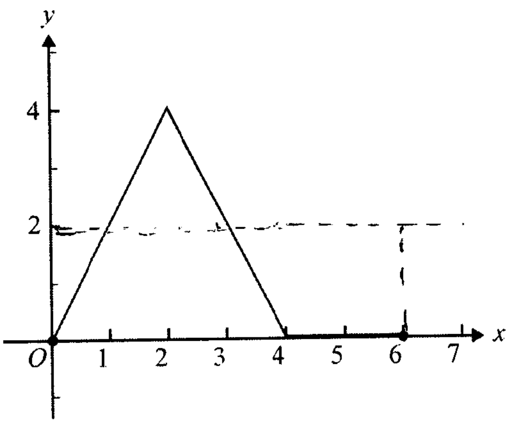
C.



D.

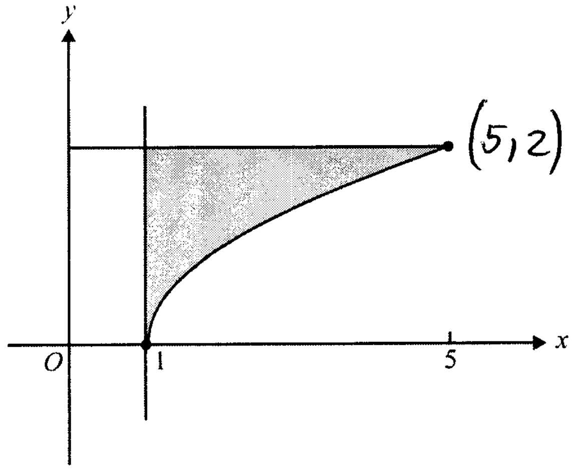


E.



Question 16

The graph of $f: [1, 5] \to R, f(x) = \sqrt{x-1}$ is shown below.



Which one of the following definite integrals could be used to find the area of the shaded region?

A. $\int_1^5 \left(\sqrt{x-1}\right)dx$

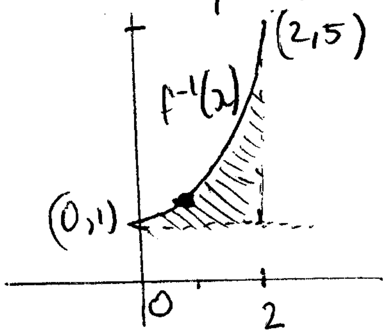
B. $\int_0^2 \left(\sqrt{x-1}\right)dx$

C. $\int_0^5 \left(2-\sqrt{x-1}\right)dx$

D. $\int_0^2 \left(x^2+1\right)dx$

E. $\int_0^2 \left(x^2\right)dx$

$\int_1^5 2-\sqrt{x-1}\ dx$ is required area, but does not appear as an option



The area is the same as the one shown in the diagram to left

$= \int_0^2 \left(f^{-1}(x)-1\right)dx$

$= \int_0^2 x^2+1-1\ dx$

$= \int_0^2 x^2\ dx$

Question 17

A and B are events of a sample space.

Given that $\Pr(A|B) = p$, $\Pr(B) = p^2$ and $\Pr(A) = p^{\frac{1}{3}}$, $\Pr(B|A)$ is equal to

A. p

B. $p^{\frac{4}{3}}$

C. $p^{\frac{7}{3}}$

D. $p^{\frac{8}{3}}$

E. p^3

$\Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)}$

$\therefore p = \frac{\Pr(A\cap B)}{p^2}$

$\therefore \Pr(A\cap B) = p^3$

$\Pr(B|A) = \frac{\Pr(B\cap A)}{\Pr(A)} = \frac{p^3}{p^{1/3}} = p^{8/3}$

Question 18

Let $g(x) = \log_2(x)$, $x > 0$.

Which one of the following equations is true for all positive real values of x?

A. $2g(8x) = g(x^2) + 8$

B. $2g(8x) = g(x^2) + 6$

C. $2g(8x) = (g(x) + 8)^2$

D. $2g(8x) = g(2x) + 6$

E. $2g(8x) = g(2x) + 64$

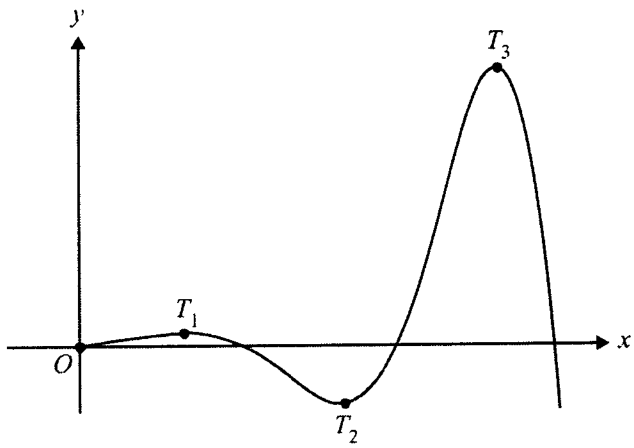
$$2g(8x) = 2\log_2(8x) = \log_2(64x^2)$$
$$= \log_2(64) + \log_2(x^2)$$
$$= 6 + \log_2(x^2)$$
$$= 6 + g(x^2)$$

TIP

Start out finding an expression for 2g(8x) since it is on the LHS of each alternative

Question 19

Part of the graph of a function $f: [0, \infty) \to R$, $f(x) = e^{x\sqrt{3}}\sin(x)$ is shown below.
The first three turning points are labelled T_1, T_2 and T_3.



The x-coordinate of T_3 is

A. $\frac{8\pi}{3}$

B. $\frac{16\pi}{3}$

C. $\frac{13\pi}{6}$

D. $\frac{17\pi}{6}$

E. $\frac{29\pi}{6}$

$$f(x) = e^{x\sqrt{3}}\sin(x)$$
$$f'(x) = e^{x\sqrt{3}}\cos x + \sqrt{3}e^{x\sqrt{3}}\sin x$$
$$= e^{x\sqrt{3}}(\cos x + \sqrt{3}\sin x)$$
$$= 0 \text{ if}$$
$$\cos x + \sqrt{3}\sin x = 0$$
$$\therefore \sqrt{3}\sin x = -\cos x$$
$$\tan x = -\frac{1}{\sqrt{3}}$$
$$\therefore x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \ldots$$

Question 20

A transformation $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ maps the graph of a function f to the graph of $y = x^2, x \in R$.

The rule of f is

A. $f(x) = -(x+5)^2$

B. $f(x) = (5-x)^2$

C. $f(x) = -(x-5)^2$

D. $f(x) = -x^2 + 5$

E. $f(x) = x^2 - 5$

$x' = x + 5$
$y' = -y$

$y' = x'^2$

$\therefore -y = (x+5)^2$
$y = -(x+5)^2$

Question 21

The cubic function $f: R \to R, f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, has no stationary points when

A. $c > \frac{b^2}{4a}$

B. $c < \frac{b^2}{4a}$

C. $c < 4b^2a$

D. $c > \frac{b^2}{3a}$

E. $c < \frac{b^2}{3a}$

$f'(x) = 3ax^2 - 2bx + c$

For stationary points, $f'(x) = 0$

$\therefore 3ax^2 - 2bx + c = 0$

For no solution, $\Delta < 0$

$\therefore (-2b)^2 - 4(3a)c < 0$

$4b^2 - 12ac < 0$

$4b^2 < 12ac$

$b^2 < 3ac$

$\therefore c > \frac{b^2}{3a}$

Question 22

Butterflies of a particular species die T days after hatching, where T is a normally distributed random variable with a mean of 120 days and a standard deviation of σ days.

If, from a population of 2000 newly hatched butterflies, 150 are expected to die in the first 90 days, then the value of σ is closest to

A. 7 days

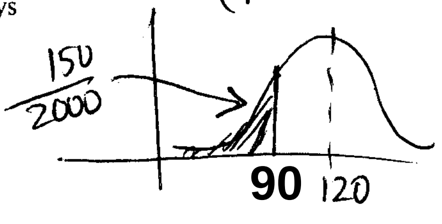
B. 13 days

C. 17 days

D. 21 days

E. 37 days

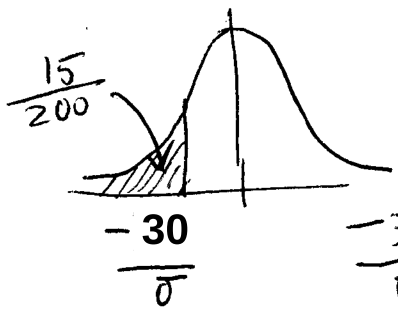
$T \stackrel{d}{=} N(\mu = 120, \sigma = ?)$



$\Pr(T < 90) = \frac{15}{200}$

$Z \stackrel{d}{=} N(0,1)$, $Z = \frac{t - 120}{\sigma}$ $\therefore \Pr\left(Z < \frac{90-120}{\sigma}\right) = \frac{15}{200}$

$\therefore \Pr\left(Z < \frac{-30}{\sigma}\right) = \frac{15}{200}$



END OF SECTION 1
TURN OVER

$\frac{-30}{\sigma} = \text{invNorm}\left(\frac{15}{200}, 0, 1\right)$

$\frac{-30}{\sigma} = -1.4395315$ $\therefore \sigma = 20.84$

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (12 marks)

Trigg the gardener is working in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature (T °C) is given by $T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right)$, $0 \le t \le 24$, where t is the time in hours from the beginning of the 24-hour time interval.

a. State the maximum temperature in the greenhouse and the values of t when this occurs. 2 marks

+2 ↑ 27
25
−2 ↓ 23

Max temp = 27

When $t = 0, 16$

b. State the period of the function T. 1 mark

$$\frac{2\pi}{\frac{\pi}{8}} = 16 \text{ hours}$$

c. Find the smallest value of t for which $T = 26$. 2 marks

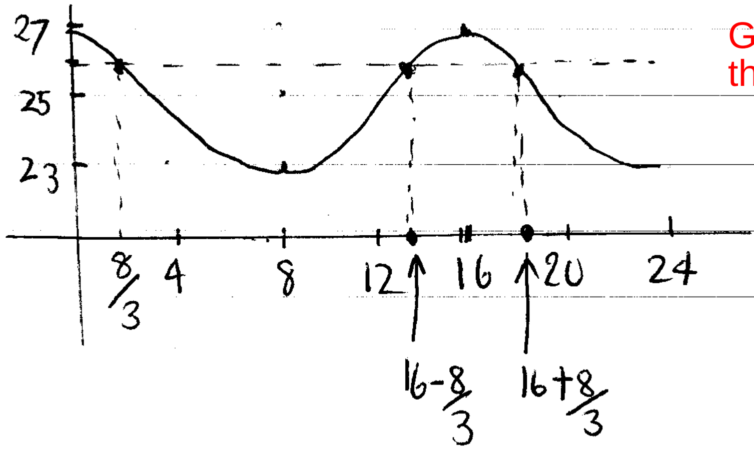
$$26 = 25 + 2\cos\left(\frac{\pi t}{8}\right)$$

$$\frac{1}{2} = \cos\left(\frac{\pi t}{8}\right)$$

$$\frac{\pi t}{8} = \frac{\pi}{3}$$

$$t = 8/3$$

d. For how many hours during the 24-hour time interval is $T \geq 26$? 2 marks



Graph assists with this problem!!

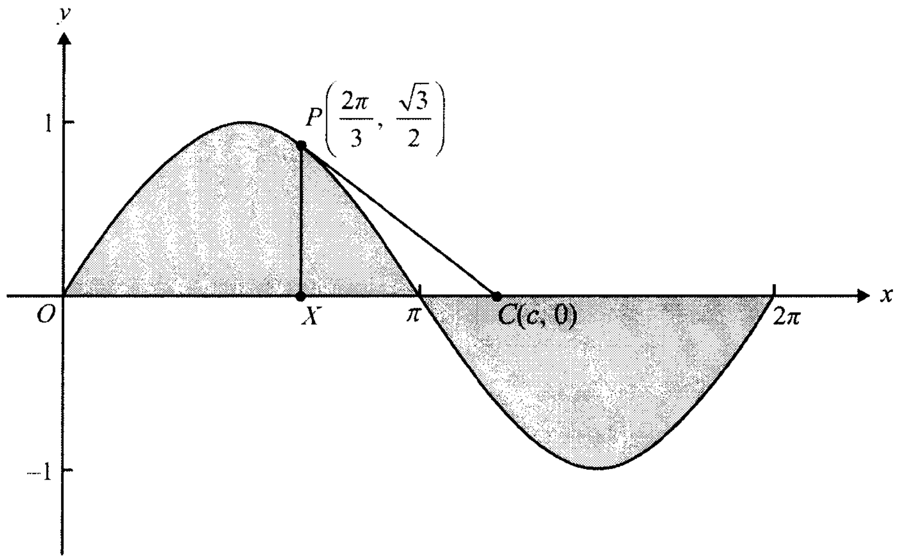
Total time for which $T \geq 26$

$$\frac{8}{3} + 2 \times \frac{8}{3}$$

$$= 8 \text{ hours}$$

Trigg is designing a garden that is to be built on flat ground. In his initial plans, he draws the graph of $y = \sin(x)$ for $0 \le x \le 2\pi$ and decides that the garden beds will have the shape of the shaded regions shown in the diagram below. He includes a garden path, which is shown as line segment PC.

The line through points $P\left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $C(c, 0)$ is a tangent to the graph of $y = \sin(x)$ at point P.



e. **i.** Find $\frac{dy}{dx}$ when $x = \frac{2\pi}{3}$. 1 mark

$$\frac{dy}{dx} = \cos x$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$$

ii. Show that the value of c is $\sqrt{3} + \frac{2\pi}{3}$. 1 mark

$$-\frac{1}{2} = \frac{\frac{\sqrt{3}}{2} - 0}{\frac{2\pi}{3} - c} \quad \text{(using gradients)}$$

$$\therefore \frac{2\pi}{3} - c = -\sqrt{3}$$

$$\therefore c = \sqrt{3} + \frac{2\pi}{3}$$

In further planning for the garden, Trigg uses a transformation of the plane defined as a dilation of factor k from the x-axis and a dilation of factor m from the y-axis, where k and m are positive real numbers.

f. Let X', P' and C' be the image, under this transformation, of the points X, P and C respectively.

i. Find the values of k and m if $X'P' = 10$ and $X'C' = 30$. 2 marks

$$(x', y') = (mx^*, ky)$$

$$\frac{X'P'}{XP} = k \quad \therefore k = \frac{10}{\frac{\sqrt{3}}{2}} = \frac{20}{\sqrt{3}} \qquad \frac{X'C'}{XC} = m \quad \therefore m = \frac{30}{\sqrt{3}}$$

$$\therefore k = \frac{20\sqrt{3}}{3}, \quad m = \frac{30}{\sqrt{3}} = 10\sqrt{3}$$

ii. Find the coordinates of the point P'. 1 mark

$$P = \left(\frac{2\pi}{3}, \frac{\sqrt{3}}{2}\right)$$

$$\therefore P' = \left(\frac{2\pi}{3} \times 10\sqrt{3}, \ \frac{\sqrt{3}}{2} \times \frac{20\sqrt{3}}{3}\right)$$

$$= \left(\frac{20\pi\sqrt{3}}{3}, \ 10\right)$$

$$P' = \left(\frac{20\sqrt{3}\pi}{3}, 10\right)$$

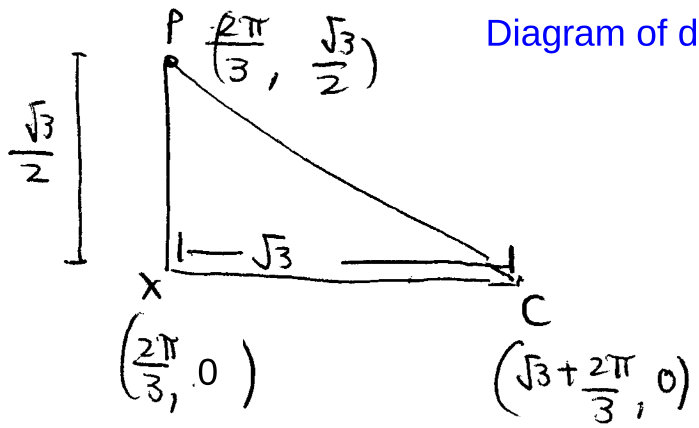


Diagram of distances helps

$XP = \frac{\sqrt{3}}{2}$

$XC = \sqrt{3}$

Question 2 (11 marks)

FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called **S**. There is a five-minute time limit on any attempt to complete **S** and if someone completes **S** in less than three minutes, they are considered fit.

a. At FullyFit's Melbourne gym, it has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member.

In a particular week, 20 members of this gym attempt **S**.

i. Find the probability, correct to four decimal places, that at least 10 of these 20 members will complete **S** in less than three minutes.

2 marks

X = no. of members who complete S in less than 3 min

$X \stackrel{d}{=} Bi\left(n = 20, p = \frac{5}{8}\right)$

$\Pr(X \geq 10) = 0.9153$

ii. Given that at least 10 of these 20 members complete **S** in less than three minutes, what is the probability, correct to three decimal places, that more than 15 of them complete **S** in less than three minutes?

3 marks

$\Pr(X > 15 \mid X \geq 10)$

$= \frac{\Pr(X > 15 \cap X \geq 10)}{\Pr(X \geq 10)}$

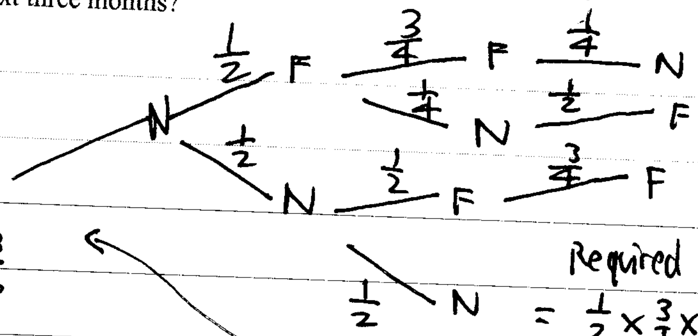
$= \frac{\Pr(X > 15)}{\Pr(X \geq 10)} = \frac{\Pr(X \geq 16)}{\Pr(X \geq 10)}$

$= 0.086$

b. Paula is a member of FullyFit's gym in San Francisco. She completes S every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

If Paula is not fit in one particular month, what is the probability that she is fit in exactly two of the next three months?

2 marks



Desired outcomes: FFN, FNF, NFF

Required probability $= \frac{1}{2}\times\frac{3}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{3}{4}$

$= \frac{3}{32} + \frac{1}{16} + \frac{3}{16}$

$= \frac{11}{32}$

c. When FullyFit surveyed all its gyms throughout the world, it was found that the time taken by members to complete S is a continuous random variable X, with a probability density function g, as defined below.

$$g(x) = \begin{cases} \dfrac{(x-3)^3+64}{256} & 1 \le x \le 3 \\ \dfrac{x+29}{128} & 3 < x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

i. Find $E(X)$, correct to four decimal places.

2 marks

$$E(X) = \int_1^3 x\left(\frac{(x-3)^3+64}{256}\right)dx + \int_3^5 x\left(\frac{x+29}{128}\right)dx$$

$\simeq 3.0458$ minutes

ii. In a random sample of 200 FullyFit members, how many members would be expected to take more than four minutes to complete S? Give your answer to the nearest integer.

2 marks

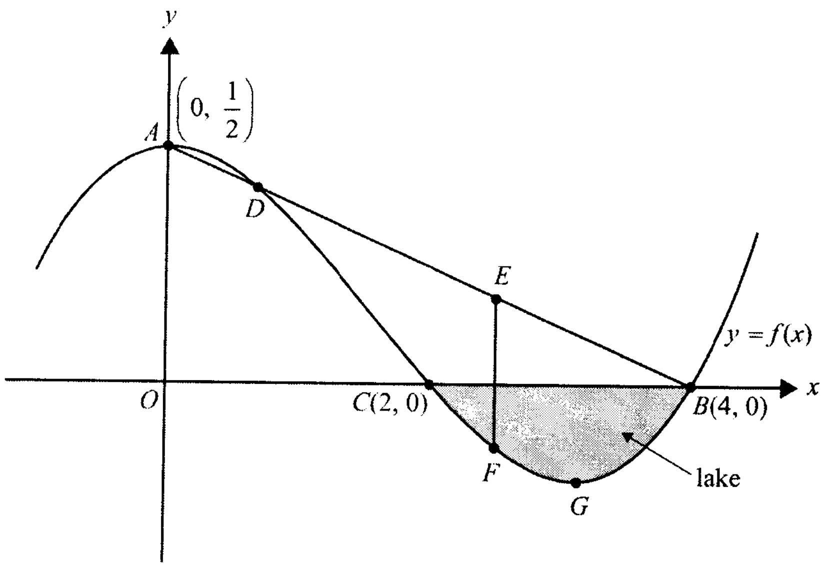
$$\Pr(X>4) = \int_4^5 \frac{x+29}{128}\,dx \simeq 0.2617$$

$200 \times 0.2617 \simeq 52.3$

Would expect 52 to complete it in more than 4 min

Question 3 (19 marks)

Tasmania Jones is in Switzerland. He is working as a construction engineer and he is developing a thrilling train ride in the mountains. He chooses a region of a mountain landscape, the cross-section of which is shown in the diagram below.



The cross-section of the mountain and the valley shown in the diagram (including a lake bed) is modelled by the function with rule

$$f(x) = \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}.$$

Tasmania knows that $A\left(0, \frac{1}{2}\right)$ is the highest point on the mountain and that $C(2, 0)$ and $B(4, 0)$ are the points at the edge of the lake, situated in the valley. All distances are measured in kilometres.

a. Find the coordinates of G, the deepest point in the lake. 3 marks

$$f'(x) = \frac{9x^2}{64} - \frac{14x}{32}$$

Let $f'(x) = 0$ $\quad \therefore \frac{9x^2}{64} - \frac{14x}{32} = 0$

$\therefore 9x^2 - 28x = 0$

$x(9x - 28) = 0$

$\therefore x = \frac{28}{9}$ (since $x > 0$)

$f\left(\frac{28}{9}\right) = \frac{-50}{243}$

$\therefore G = \left(\frac{28}{9}, \frac{-50}{243}\right)$

Exact co-ordinates required

Tasmania's train ride is made by constructing a straight railway line AB from the top of the mountain, A, to the edge of the lake, B. The section of the railway line from A to D passes through a tunnel in the mountain.

b. Write down the equation of the line that passes through A and B. 2 marks

$$m = \frac{\frac{1}{2} - 0}{0 - 4} = -\frac{1}{8}$$

$$\therefore y = -\frac{1}{8}x + \frac{1}{2}$$

c. i. Show that the x-coordinate of D, the end point of the tunnel, is $\frac{2}{3}$. 1 mark

D is the intersection point of $y = \frac{-1}{8}x + \frac{1}{2}$ and $f(x) = \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}$,

When $x = \frac{2}{3}$, substitute into $y = \frac{-1}{8}x + \frac{1}{2} = \frac{-1}{8} \times \frac{2}{3} + \frac{1}{2} = \frac{5}{12}$

$f\left(\frac{2}{3}\right) = \frac{3}{64} \times \left(\frac{2}{3}\right)^3 - \frac{7}{32} \times \left(\frac{2}{3}\right)^2 + \frac{1}{2} = \frac{5}{12}$ $\therefore$ x co-ordinate at D is $\frac{2}{3}$.

Since this is a "show that" question, reasoning is required.

ii. Find the length of the tunnel AD. 2 marks

$D = \left(\frac{2}{3}, \frac{5}{12}\right)$

$A = \left(0, \frac{1}{2}\right)$

$$\text{Distance} = \sqrt{\left(\frac{2}{3} - 0\right)^2 + \left(\frac{5}{12} - \frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \left(\frac{-1}{12}\right)^2} = \sqrt{\frac{4}{9} + \frac{1}{144}}$$

$$= \frac{\sqrt{65}}{12} \text{ km}$$

Exact value required

In order to ensure that the section of the railway line from D to B remains stable, Tasmania constructs vertical columns from the lake bed to the railway line. The column EF is the longest of all possible columns. (Refer to the diagram on page 18.)

d. **i.** Find the x-coordinate of E. 2 marks

Vertical separation : $u(x) = \frac{-x}{8} + \frac{1}{2} - f(x)$, $\frac{2}{3} < x < 4$

For a maximum, $u'(x) = 0$

Solving: $x = \frac{2(\sqrt{31} + 7)}{9}$ (since $\frac{2}{3} < x < 4$)

ii. Find the length of the column EF in metres, correct to the nearest metre. 2 marks

$u\left(\frac{2\sqrt{31} + 14}{9}\right) \simeq 0.33600$

$\simeq 336$ m

Tasmania's train travels down the railway line from A to B. The speed, in km/h, of the train as it moves down the railway line is described by the function

$$V: [0, 4] \to R, V(x) = k\sqrt{x} - mx^2,$$

where x is the x-coordinate of a point on the front of the train as it moves down the railway line, and k and m are positive real constants.

The train begins its journey at $A\left(0, \frac{1}{2}\right)$. It increases its speed as it travels down the railway line.

The train then slows to a stop at $B(4, 0)$, that is $V(4) = 0$.

e. Find k in terms of m. 1 mark

$V(4) = 0$

$\therefore k\sqrt{4} - m \times 4^2 = 0$

$2k - 16m = 0$

$k = 8m$

f. Find the value of x for which the speed, V, is a maximum.

2 marks

$$V(x) = 8m\sqrt{x} - mx^2$$
$$V'(x) = 4mx^{-1/2} - 2mx$$
$$= m\left(\frac{4}{\sqrt{x}} - 2x\right)$$

Let $V'(x) = 0$

$$\therefore \frac{4}{\sqrt{x}} - 2x = 0 \quad \therefore x = 2^{\frac{2}{3}}$$

Tasmania is able to change the value of m on any particular day. As m changes, the relationship between k and m remains the same.

g. If, on one particular day, $m = 10$, find the maximum speed of the train, correct to one decimal place.

2 marks

$$V(x) = 8m\sqrt{x} - mx^2$$
$$= m(8\sqrt{x} - x^2)$$

$$V_{max} = V\left(2^{\frac{2}{3}}\right) = 6m\,2^{1/3}$$

If $m = 10$, $V_{max} = 6 \times 10 \times 2^{1/3} \simeq 75.6$ km/hr

h. If, on another day, the maximum value of V is 120, find the value of m.

2 marks

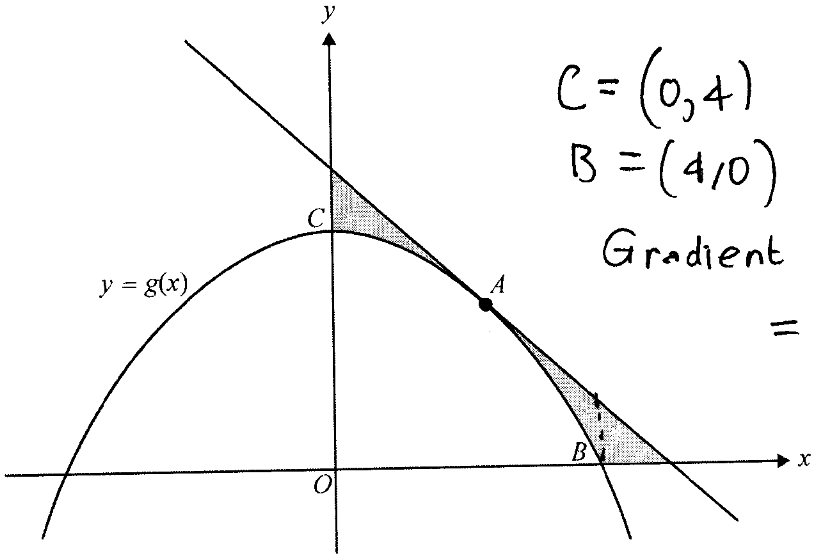
$$120 = 6m \times 2^{1/3}$$
$$20 = m \times 2^{1/3}$$

$$m = \frac{20}{2^{1/3}} = 10 \times 2^{2/3}$$

Exact value required

Question 4 (16 marks)

Part of the graph of a function $g: R \rightarrow R$, $g(x) = \dfrac{16 - x^2}{4}$ is shown below.



$C = (0, 4)$

$B = (4, 0)$

Gradient BC $= \dfrac{4 - 0}{0 - 4}$ $= -1$

a. Points B and C are the positive x-intercept and y-intercept of the graph of g, respectively, as shown in the diagram above. The tangent to the graph of g at the point A is parallel to the line segment BC.

i. Find the equation of the tangent to the graph of g at the point A. 2 marks

$\dfrac{dy}{dx}$ at A $= -1$

$\therefore \dfrac{dy}{dx} = -1$

$-\dfrac{x}{2} = -1$

$\therefore x = 2$

$g(2) = \dfrac{16 - 2^2}{4} = 3$

$A = (2, 3)$, $m = -1$

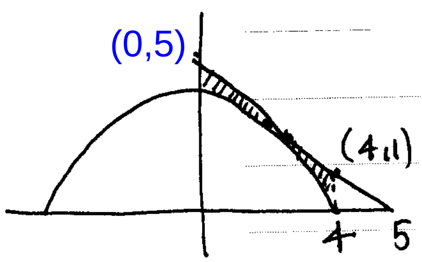
$y - 3 = -(x - 2)$

$y = -x + 5$

ii. The shaded region shown in the diagram above is bounded by the graph of g, the tangent at the point A, and the x-axis and y-axis.

Evaluate the area of this shaded region. 3 marks

Required area



$$= \int_0^4 \left(-x + 5 - g(x)\right) dx + \int_4^5 (-x + 5)\, dx$$

$$= \int_0^4 \left(-x + 5 - g(x)\right) dx + \frac{1}{2}$$

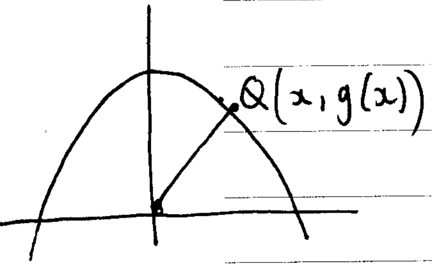
$$= \frac{4}{3} + \frac{1}{2} = \frac{11}{6} \text{ sq. units}$$

$\int_4^5 (-x + 5)\, dx$ is the area of a right triangle with height 1 and base 1 unit $= \dfrac{1 \times 1}{2}$

NOTE: An alternative and possibly easier way of solving this was to calculate the area of the large right triangle (base 5 units and height 5 units) and then subtract the area under the curve.

b. Let Q be a point on the graph of $y = g(x)$.
Find the positive value of the x-coordinate of Q, for which the distance OQ is a minimum and find the minimum distance. 3 marks

$$d(x) = \sqrt{x^2 + [g(x)]^2}$$

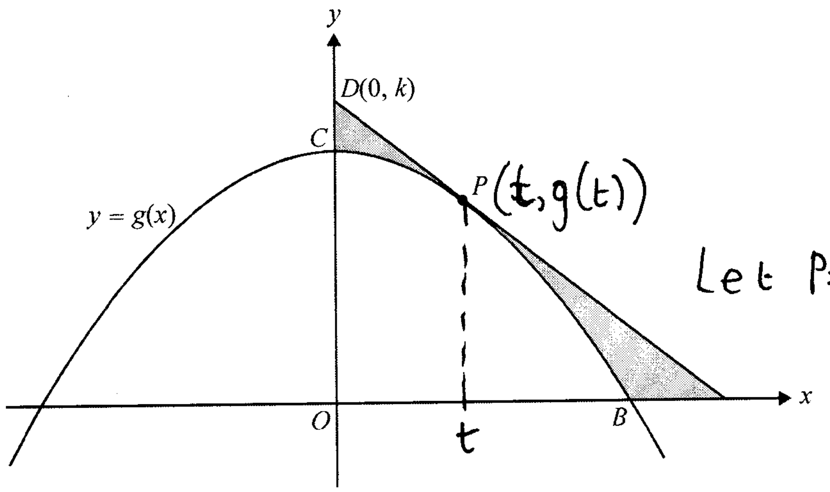


For a minimum, $d'(x) = 0$

Solving: $x = 2\sqrt{2}$ (since $x > 0$)

$\therefore d_{min} = d(2\sqrt{2}) = 2\sqrt{3}$

The tangent to the graph of g at a point P has a **negative** gradient and intersects the y-axis at point $D(0, k)$, where $5 \le k \le 8$.



Let $P = \left(t, \frac{16-t^2}{4}\right)$

c. Find the gradient of the tangent in terms of k. 2 marks

At P, $\left.\frac{dy}{dx}\right|_{x=t} = -\frac{t}{2}$

But gradient of tangent $= \dfrac{k - \left(\frac{16-t^2}{4}\right)}{0 - t}$

$$\frac{k - \left(\frac{16-t^2}{4}\right)}{-t} = \frac{-t}{2}$$

$$k - 4 + \frac{t^2}{4} = \frac{t^2}{2}$$

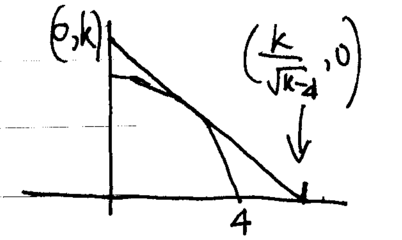
$\therefore k - 4 = \frac{t^2}{4}$ $\therefore t = 2\sqrt{k-4}$ $\therefore \frac{-t}{2} = -\sqrt{k-4}$

d. **i.** Find the rule $A(k)$ for the function of k that gives the area of the shaded region. 2 marks

$A(k) = A_{triangle} - \int_0^4 g(x)\,dx$

$= \frac{1}{2} \cdot k \cdot \frac{k}{\sqrt{k-4}} - \frac{32}{3}$

$A(k) = \frac{k^2}{2\sqrt{k-4}} - \frac{32}{3}$ sq. units



$y = -\sqrt{k-4}\,x + k$

x-int: $\frac{k}{\sqrt{k-4}}$

ii. Find the **maximum** area of the shaded region and the value of k for which this occurs. 2 marks

$A(k) = \frac{k^2}{2\sqrt{k-4}} - \frac{32}{3}$, $5 \le k \le 8$.

From graph of $a(k) = \frac{k^2}{2\sqrt{k-4}} - \frac{32}{3}$, the maximum occurs at an endpoint.

$A(5) = \frac{25}{2} - \frac{32}{3} = \frac{11}{6}$

$A(8) = \frac{16}{3}$ $\therefore A_{max} = \frac{16}{3}$

(Check both endpoints)

iii. Find the **minimum** area of the shaded region and the value of k for which this occurs. 2 marks

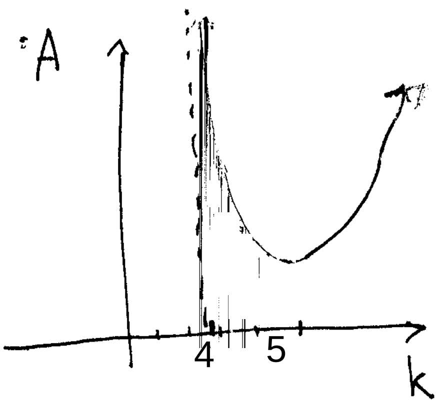
Minimum occurs at a stationary point

$A'(k) = 0$

$\therefore k = \frac{16}{3}$

$A\left(\frac{16}{3}\right) = \frac{64\sqrt{3}}{9} - \frac{32}{3}$

Minimum area $= \frac{64\sqrt{3}}{9} - \frac{32}{3}$ sq. units



Graph helps in visualizing where the minimum is and the endpoints