Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (5 marks)

a. If $y = x^2 \sin(x)$, find $\frac{dy}{dx}$. 2 marks

Let $u = x^2$ $\quad v = \sin x$

$u' = 2x$ $\quad v' = \cos x$

$\frac{dy}{dx} = vu' + uv'$

$\therefore \frac{dy}{dx} = 2x \sin x + x^2 \cos x$

b. If $f(x) = \sqrt{x^2 + 3}$, find $f'(1)$. 3 marks

$f(x) = (x^2+3)^{\frac{1}{2}}$

$\therefore f'(x) = \frac{1}{2}(x^2+3)^{-\frac{1}{2}} \times 2x$

$= \frac{x}{\sqrt{x^2+3}}$

$\therefore f'(1) = \frac{1}{\sqrt{1^2+3}}$

$= \frac{1}{\sqrt{4}}$

$= \frac{1}{2}$

Question 2 (2 marks)

Let $\int_4^5 \frac{2}{2x-1}dx = \log_e(b)$.

Find the value of b.

$$\int_4^5 \frac{2}{2x-1}\,dx = \left[\frac{2}{2}\log_e|2x-1|\right]_4^5$$
$$= \left[\log_e|2x-1|\right]_4^5$$
$$= \log_e 9 - \log_e 7$$
$$= \log_e\left(\frac{9}{7}\right) \qquad \therefore b = \frac{9}{7}$$

Question 3 (2 marks)

Solve $2\cos(2x) = -\sqrt{3}$ for x, where $0 \le x \le \pi$.

$$\cos(2x) = \frac{-\sqrt{3}}{2} \qquad 0 \le 2x \le 2\pi$$
$$\therefore 2x = \frac{5\pi}{6}, \frac{7\pi}{6}$$
$$x = \frac{5\pi}{12}, \frac{7\pi}{12}$$

Question 4 (2 marks)

Solve the equation $2^{3x-3} = 8^{2-x}$ for x.

$$2^{3x-3} = \left(2^3\right)^{2-x}$$
$$\therefore 2^{3x-3} = 2^{6-3x}$$
$$\therefore 3x-3 = 6-3x$$
$$6x = 9$$
$$x = \frac{3}{2}$$

Question 5 (7 marks)

Consider the function $f:[-1,3]\to R,\ f(x)=3x^2-x^3$.

a. Find the coordinates of the stationary points of the function. 2 marks

$f'(x) = 6x - 3x^2$

For a stationary point, $f'(x) = 0$

$\therefore 3x(2-x) = 0$

$x = 0, 2$

$f(0) = 0$ and $f(2) = 3\times 2^2 - 2^3 = 4$

$\therefore$ Stationary points: $(0,0)$, $(2,4)$

b. On the axes below, sketch the graph of f.
Label any end points with their coordinates. 2 marks

$f(-1) = 3(-1)^2 - (-1)^3$

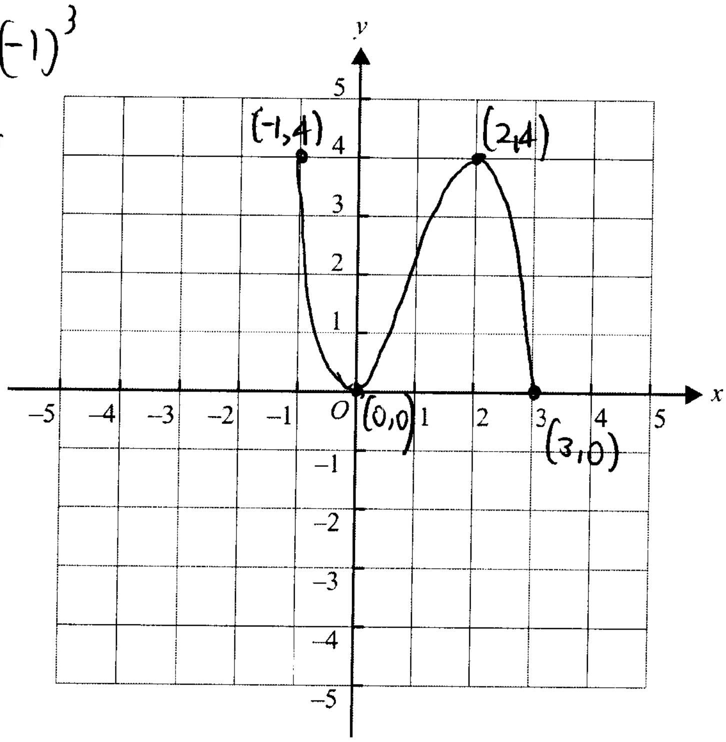
$= 3 + 1 = 4$

$\therefore (-1, 4)$

$f(3) = 3\times 3^2 - 3^3$

$= 0.$

$\therefore (3,0)$



c. Find the area enclosed by the graph of the function and the horizontal line given by $y = 4$. 3 marks

$$\int_{-1}^{2} \left(4 - (3x^2 - x^3)\right) dx$$

$$= \int_{-1}^{2} (x^3 - 3x^2 + 4)\, dx$$

$$= \left[\frac{x^4}{4} - x^3 + 4x\right]_{-1}^{2}$$

$$= \left(\frac{2^4}{4} - 2^3 + 2\times 4\right) - \left(\frac{1}{4} + 1 - 4\right)$$

$$= (4 - 8 + 8) - \left(\frac{-11}{4}\right)$$

$$= 6\frac{3}{4} \text{ sq. units}$$

$$= \frac{27}{4} \text{ sq. units}$$

Question 6 (2 marks)

Solve $\log_e(x) - 3 = \log_e\left(\sqrt{x}\right)$ for x, where $x > 0$.

$$\log_e x - 3 = \log_e \sqrt{x}$$

$$\therefore \log_e x - 3 = \frac{1}{2}\log_e x$$

$$\therefore \frac{1}{2}\log_e x = 3$$

$$\log_e x = 6 \quad \therefore x = e^6$$

Question 7 (3 marks)

If $f'(x) = 2\cos(x) - \sin(2x)$ and $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$, find $f(x)$.

$$f'(x) = 2\cos x - \sin 2x$$

$$\therefore f(x) = \int 2\cos x - \sin 2x \, dx$$

$$f(x) = 2\sin x + \frac{1}{2}\cos 2x + c$$

When $x = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = \frac{1}{2}$

$$\therefore \frac{1}{2} = 2\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos(\pi) + c$$

$$\therefore \frac{1}{2} = 2 - \frac{1}{2} + c$$

$$\therefore -1 = c$$

$$\therefore f(x) = 2\sin x + \frac{1}{2}\cos 2x - 1$$

Question 8 (4 marks)

A continuous random variable, X, has a probability density function given by

$$f(x)=\begin{cases}\frac{1}{5}e^{-\frac{x}{5}} & x\geq 0\\ 0 & x<0\end{cases}$$

The median of X is m.

a. Determine the value of m.

2 marks

$\Pr(X < m) = 0.5 \quad \therefore \int_0^m \frac{1}{5}e^{-\frac{x}{5}}\,dx = \frac{1}{2}$

$\therefore \left[-e^{-\frac{x}{5}}\right]_0^m = \frac{1}{2}$

$\therefore -e^{-\frac{m}{5}} + e^0 = \frac{1}{2}$

$\therefore -e^{-m/5} = -\frac{1}{2}$

$e^{-m/5} = \frac{1}{2}$

$-\frac{m}{5} = \log_e\left(\frac{1}{2}\right) \quad \therefore \frac{m}{5} = \log_e 2$

$m = 5\log_e 2$

b. The value of m is a number greater than 1.

Find $\Pr(X<1|X\leq m)$.

2 marks

$\Pr(X<1|X\leq m) = \dfrac{\Pr(X<1 \cap X\leq m)}{\Pr(X\leq m)} = \dfrac{\Pr(X<1)}{\frac{1}{2}}$

$= 2\Pr(X<1)$

$= 2\int_0^1 \frac{1}{5}e^{-x/5}\,dx$

$= 2\left[-e^{-x/5}\right]_0^1$

$= 2\left(1-e^{-1/5}\right)$

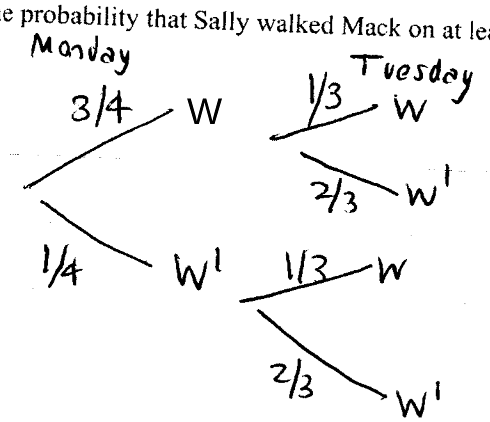
Question 9 (6 marks)

Sally aims to walk her dog, Mack, most mornings. If the weather is pleasant, the probability that she will walk Mack is $\frac{3}{4}$, and if the weather is unpleasant, the probability that she will walk Mack is $\frac{1}{3}$.

Assume that pleasant weather on any morning is independent of pleasant weather on any other morning.

a. In a particular week, the weather was pleasant on Monday morning and unpleasant on Tuesday morning.

Find the probability that Sally walked Mack on at least one of these two mornings. 2 marks



Required probability

$$= 1 - \Pr(W' \cap W')$$

$$= 1 - \frac{1}{4} \times \frac{2}{3}$$

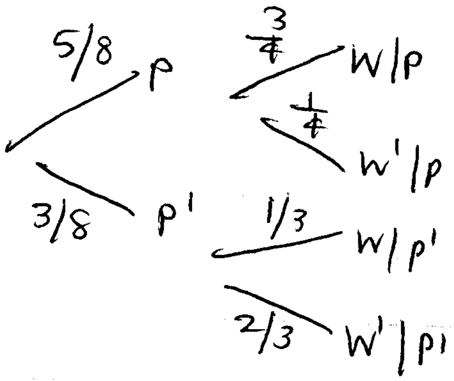
$$= 1 - \frac{1}{6}$$

$$= \frac{5}{6}$$

Tree Diagram is very helpful. Note that Monday and Tuesday go across because Monday occurs and then Tuesday occurs as successive independent events

b. In the month of April, the probability of pleasant weather in the morning was $\frac{5}{8}$.

i. Find the probability that on a particular morning in April, Sally walked Mack. 2 marks



Again a Tree Diagram is very helpful. This time, the Tree Diagram applies to a single day. On this day, she either walks the dog or does not, and the probability of walking depends on the weather (pleasant **OR** not pleasant

$$\Pr(W) = \Pr(P \cap W) + \Pr(P' \cap W)$$

$$= \frac{5}{8} \times \frac{3}{4} + \frac{3}{8} \times \frac{1}{3} = \frac{15}{32} + \frac{1}{8} = \frac{19}{32}$$

ii. Using your answer from **part b.i.**, or otherwise, find the probability that on a particular morning in April, the weather was pleasant, given that Sally walked Mack that morning.

2 marks

Find : $\Pr(P \mid W)$

$$= \frac{\Pr(P \cap W)}{\Pr(W)}$$

$$= \frac{\frac{5}{8} \times \frac{3}{4}}{\frac{19}{32}}$$

$$= \frac{\frac{15}{32}}{\frac{19}{32}}$$

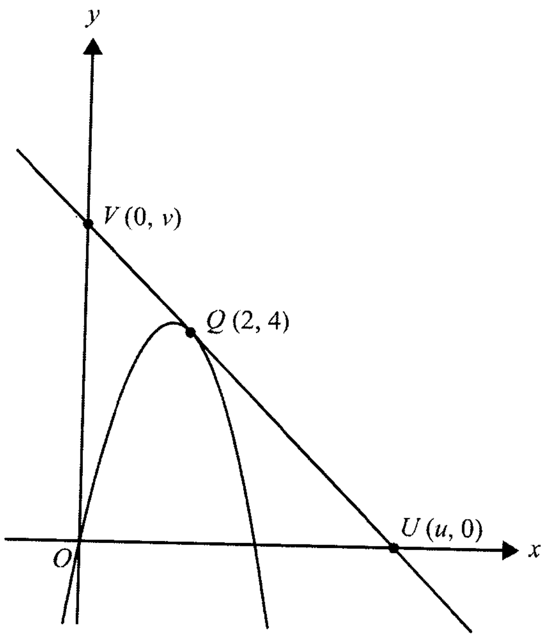
$$= \frac{15}{19}$$

TURN OVER

Question 10 (7 marks)

A line intersects the coordinate axes at the points U and V with coordinates $(u, 0)$ and $(0, v)$, respectively, where u and v are positive real numbers and $\frac{5}{2} \le u \le 6$.

a. When $u = 6$, the line is a tangent to the graph of $y = ax^2 + bx$ at the point Q with coordinates $(2, 4)$, as shown.



If a and b are non-zero real numbers, find the values of a and b. 3 marks

Since $(2,4)$ lies on curve:

$4 = a(2)^2 + b(2)$

$\therefore 4 = 4a + 2b$

$\therefore 2 = 2a + b$ (1)

Since $u = 6$, $U = (6,0)$

$\therefore$ Gradient of tangent $= \frac{4-0}{2-6} = \frac{4}{-4} = -1$

$\therefore f'(2) = -1$

$f'(x) = 2ax + b$

$\therefore -1 = 4a + b$ (2)

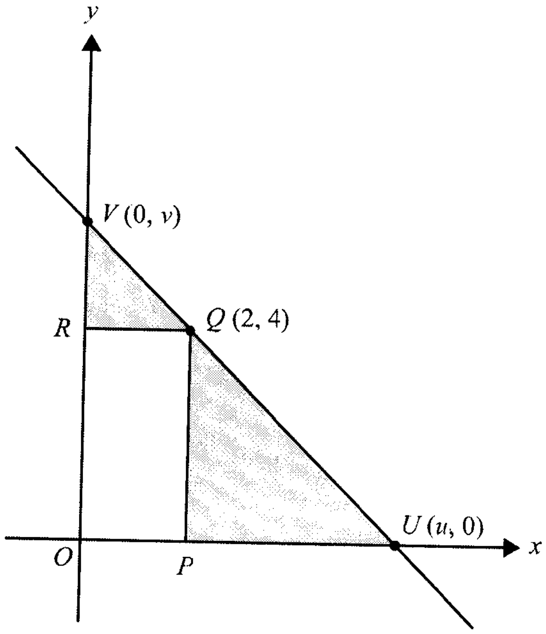
(1) − (2): $3 = -2a \therefore a = \frac{-3}{2}$

Sub. for a in (1):

$2 = 2 \times \frac{-3}{2} + b \quad \therefore b = 5$

Question 10 – continued

b. The rectangle *OPQR* has a vertex at *Q* on the line. The coordinates of *Q* are (2, 4), as shown.



i. Find an expression for v in terms of u. 1 mark

$$\frac{v-4}{0-2} = \frac{4-0}{2-u}$$

$$\therefore \frac{v-4}{-2} = \frac{4}{2-u}$$

$$\therefore \frac{v-4}{2} = \frac{4}{u-2}$$

Easiest way is to use the fact that gradient from V to Q is equal to gradient from Q to U.

$$\therefore v-4 = \frac{8}{u-2}$$

$$v = \frac{8}{u-2} + 4$$

$$\text{or} \quad v = \frac{4u}{u-2}$$

ii. Find the **minimum** total shaded area and the value of u for which the area is a minimum.

2 marks

Shaded area $A = A_{triangle} - A_{rectangle}$

$$= \frac{vu}{2} - 2\times 4$$

$$= \frac{u}{2} \times \frac{4u}{u-2} - 8$$

$$= \frac{2u^2}{u-2} - 8 \quad , \quad \frac{5}{2} \le u \le 6$$

$$\frac{dA}{du} = \frac{(u-2).4u - 2u^2 \times 1}{(u-2)^2} = \frac{4u^2 - 8u - 2u^2}{(u-2)^2}$$

$$= \frac{2u^2 - 8u}{(u-2)^2}$$

iii. Find the **maximum** total shaded area and the value of u for which the area is a maximum.

1 mark

Endpoints

$$A\left(\frac{5}{2}\right) = \frac{2\times\left(\frac{5}{2}\right)^2}{\frac{5}{2}-2} - 8 = \frac{2\times\frac{25}{4}}{\frac{1}{2}} - 8 = 17$$

$$A(6) = \frac{2\times 6^2}{4} - 8 = \frac{72}{4} - 8 = 10$$

$\therefore$ Maximum area $= 17$ sq. units when $u = \frac{5}{2}$

For a stationary point, $\frac{dA}{du} = 0$ $\therefore$ $2u^2 - 8u = 0$

$$2u(u-4) = 0$$

$$u = 4 \quad \left(\text{since } u \in \left[\frac{5}{2}, 6\right]\right)$$

$$A(4) = \frac{2\times 4^2}{(4-2)} - 8 = 8$$

Minimum area $= 8$ sq. units

when $u = 4$.

If sketched graph of A versus u would look like this