Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (4 marks)

a. Let $y = (5x + 1)^7$.

Find $\frac{dy}{dx}$. 1 mark

$$\frac{dy}{dx} = 7(5x+1)^6 \times 5$$
$$= 35(5x+1)^6$$

b. Let $f(x) = \frac{\log_e(x)}{x^2}$.

i. Find $f'(x)$. 2 marks

Let $u = \log_e x$, $V = x^2$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$
$$= \frac{x^2 \cdot \frac{1}{x} - 2x\log_e x}{x^4}$$
$$= \frac{x - 2x\log_e x}{x^4}$$
$$= \frac{1 - 2\log_e x}{x^3}$$

ii. Evaluate $f'(1)$. 1 mark

$$f'(1) = \frac{1 - 2\log_e 1}{1^3}$$
$$= 1$$

Question 2 (3 marks)

Let $f'(x) = 1 - \frac{3}{x}$, where $x \neq 0$.

Given that $f(e) = -2$, find $f(x)$.

$f(x) = \int 1 - \frac{3}{x} \, dx$

$\therefore f(x) = x - 3\log_e x + c$

When $x = e$, $y = -2$

$\therefore -2 = e - 3\log_e e + c$

$\therefore -2 = e - 3 + c$

$\therefore c = 1 - e$

$\therefore f(x) = x - 3\log_e x + 1 - e$, where $x > 0$

Question 3 (2 marks)

Evaluate $\int_1^4 \left(\frac{1}{\sqrt{x}}\right) dx$.

$$\int_1^4 x^{-1/2}\, dx$$

$$= \left[\frac{x^{1/2}}{1/2}\right]_1^4$$

$$= \left[2\sqrt{x}\right]_1^4$$

$$= 2\sqrt{4} - 2\sqrt{1}$$

$$= 4 - 2$$

$$= 2$$

Question 4 (6 marks)

Consider the function $f:[-3,\ 2] \to R,\ f(x) = \frac{1}{2}\left(x^3 + 3x^2 - 4\right)$.

a. Find the coordinates of the stationary points of the function. 2 marks

$f'(x) = \frac{1}{2}(3x^2 + 6x)$

For a stationary point, $f'(x) = 0$

$\therefore 3x^2 + 6x = 0$

$3x(x+2) = 0$

$x = 0, -2$

If $x = 0$, $f(0) = -2$

If $x = -2$, $f(-2) = \frac{1}{2}(-8 + 12 - 4)$

$= 0$

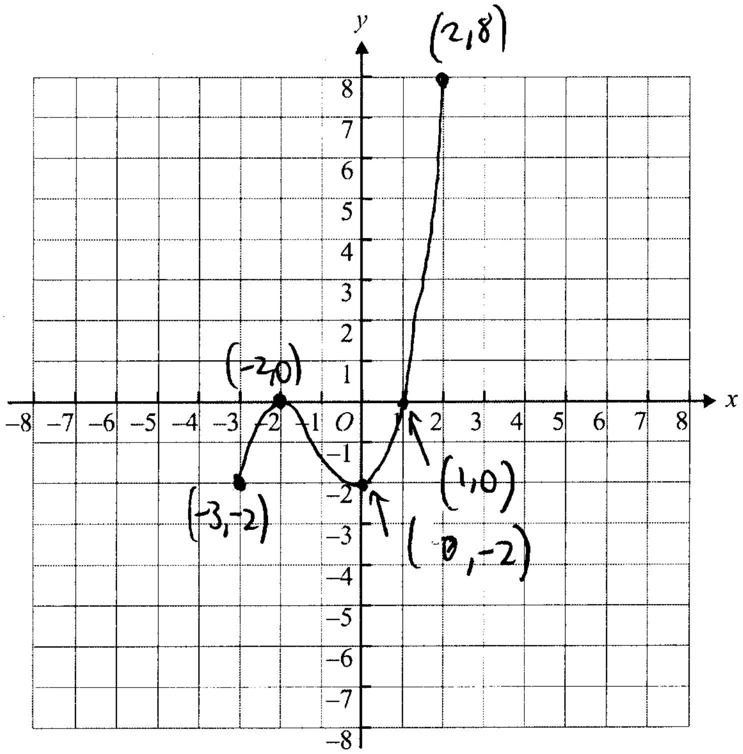
$\therefore (0, -2)$ and $(-2, 0)$

are stationary points.

The rule for f can also be expressed as $f(x)=\frac{1}{2}(x-1)(x+2)^2$. Domain : $[-3, 2]$

b. On the axes below, sketch the graph of f, clearly indicating axis intercepts and turning points. Label the end points with their coordinates. 2 marks

$f(-3)=\frac{1}{2}\times -4\times 1$
$=-2$
$f(2)=\frac{1}{2}\times 1\times 16$
$=8$



c. Find the average value of f over the interval $0 \le x \le 2$. 2 marks

$$\frac{1}{2-0}\int_0^2 \frac{1}{2}\left(x^3+3x^2-4\right)dx$$

$$=\frac{1}{4}\int_0^2 x^3+3x^2-4\,dx$$

$$=\frac{1}{4}\left[\frac{x^4}{4}+x^3-4x\right]_0^2$$

$$=\frac{1}{4}\left(\frac{16}{4}+8-8\right)-\frac{1}{4}(0)$$

$$=\frac{1}{4}\times 4$$

$$=1$$

Question 5 (3 marks)

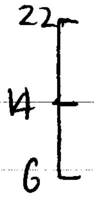
On any given day, the depth of water in a river is modelled by the function

$$h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right), \quad 0 \le t \le 24$$

where h is the depth of water, in metres, and t is the time, in hours, after 6 am.

a. Find the minimum depth of the water in the river. 1 mark

6 metres



b. Find the values of t for which $h(t) = 10$. 2 marks

$$10 = 14 + 8\sin\left(\frac{\pi t}{12}\right), \quad 0 \le t \le 24$$

$$-\frac{1}{2} = \sin\left(\frac{\pi t}{12}\right), \quad 0 \le \frac{\pi t}{12} \le 2\pi$$

$$\therefore \frac{\pi t}{12} = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{t}{12} = \frac{7}{6}, \frac{11}{6}$$

$$\therefore t = 14, 22$$

Question 7 (5 marks)

a. Solve $\log_2(6 - x) - \log_2(4 - x) = 2$ for x, where $x < 4$. 2 marks

$$\log_2\left(\frac{6-x}{4-x}\right) = 2$$

$$\frac{6-x}{4-x} = 2^2$$

$$\frac{6-x}{4-x} = 4$$

$$\therefore 6-x = 16-4x$$

$$3x = 10$$

$$x = \frac{10}{3}$$

b. Solve $3e^t = 5 + 8e^{-t}$ for t. 3 marks

$$3e^t = 5 + \frac{8}{e^t}$$

Let $e^t = p$

$$3p = 5 + \frac{8}{p}$$

$$3p^2 = 5p + 8$$

$$3p^2 - 5p - 8 = 0$$

$$3p^2 + 3p - 8p - 8 = 0$$

$$3p(p+1) - 8(p+1) = 0$$

$$(p+1)(3p-8) = 0$$

$$p = -1, \frac{8}{3}$$

But $p = e^t$

$e^t = -1$ No solution

$e^t = \frac{8}{3}$ $\therefore t = \log_e\left(\frac{8}{3}\right)$

Question 8 (3 marks)

For events A and B from a sample space, $\Pr(A|B) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$.

a. Calculate $\Pr(A \cap B)$.

1 mark

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

$\therefore \frac{3}{4} = \frac{\Pr(A \cap B)}{\frac{1}{3}}$

$\therefore \Pr(A \cap B) = \frac{1}{4}$

b. Calculate $\Pr(A' \cap B)$, where A' denotes the complement of A.

1 mark

	B	B'
A	$\frac{1}{4}$	
A'		
	$\frac{1}{3}$	

$\Pr(A' \cap B) = \frac{1}{3} - \frac{1}{4}$

$= \frac{1}{12}$

c. If events A and B are independent, calculate $\Pr(A \cup B)$.

1 mark

If A and B are independent,

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$\therefore \frac{1}{4} = \Pr(A) \times \frac{1}{3}$

$\therefore \Pr(A) = \frac{3}{4}$

[or simply, if A and B are independent, $\Pr(A|B) = \Pr(A) = \frac{3}{4}$]

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\therefore \Pr(A \cup B) = \frac{3}{4} + \frac{1}{3} - \frac{1}{4}$

$= \frac{5}{6}$

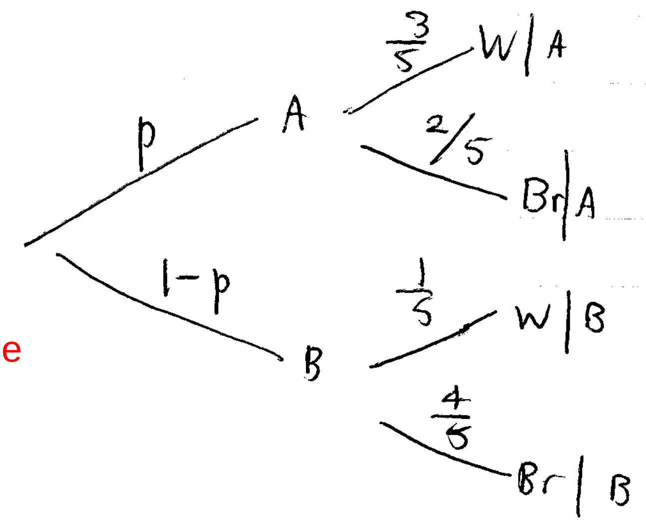
Question 9 (4 marks)

An egg marketing company buys its eggs from farm A and farm B. Let p be the proportion of eggs that the company buys from farm A. The rest of the company's eggs come from farm B. Each day, the eggs from both farms are taken to the company's warehouse.

Assume that $\frac{3}{5}$ of all eggs from farm A have white eggshells and $\frac{1}{5}$ of all eggs from farm B have white eggshells.

a. An egg is selected at random from the set of all eggs at the warehouse.

Find, in terms of p, the probability that the egg has a white eggshell. 1 mark



Drawing a Tree Diagram is very helpful.

$$\Pr(W) = \Pr(A) \times \Pr(W|A) + \Pr(B) \times \Pr(W|B)$$

$$= p \times \frac{3}{5} + (1-p) \times \frac{1}{5}$$

$$= \frac{3p}{5} + \frac{1}{5} - \frac{p}{5}$$

$$= \frac{2p}{5} + \frac{1}{5}$$

Question 9 – continued

b. Another egg is selected at random from the set of all eggs at the warehouse.

i. Given that the egg has a white eggshell, find, in terms of p, the probability that it came from farm B.

2 marks

$$\Pr(B|W) = \frac{\Pr(B \cap W)}{\Pr(W)} = \frac{\frac{1}{5}(1-p)}{\frac{(2p+1)}{5}}$$

$$= \frac{1-p}{2p+1}$$

ii. If the probability that this egg came from farm B is 0.3, find the value of p.

1 mark

$$\frac{3}{10} = \frac{1-p}{2p+1}$$

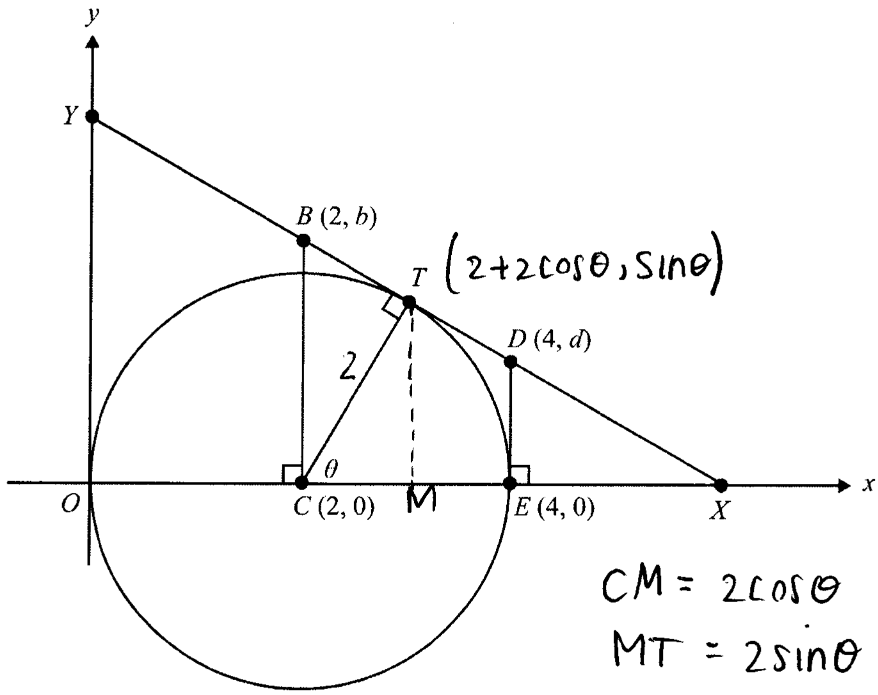
$$\therefore \quad 6p + 3 = 10 - 10p$$

$$16p = 7$$

$$p = \frac{7}{16}$$

Question 10 (7 marks)

The diagram below shows a point, T, on a circle. The circle has radius 2 and centre at the point C with coordinates $(2, 0)$. The angle ECT is θ, where $0 < \theta \leq \frac{\pi}{2}$.



The diagram also shows the tangent to the circle at T. This tangent is perpendicular to CT and intersects the x-axis at point X and the y-axis at point Y.

a. Find the coordinates of T in terms of θ. 1 mark

$(2 + 2\cos\theta, \; 2\sin\theta)$

b. Find the gradient of the tangent to the circle at T in terms of θ. 1 mark

Gradient of CT $= \dfrac{2\sin\theta - 0}{2 + 2\cos\theta - 2} = \dfrac{2\sin\theta}{2\cos\theta} = \tan\theta$

$\therefore$ Gradient of tangent $= -\dfrac{1}{m_{CT}}$

$= -\dfrac{1}{\tan\theta}$

Question 10 – continued

c. The equation of the tangent to the circle at T can be expressed as

$$\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$$

i. Point B, with coordinates $(2, b)$, is on the line segment XY.

Find b in terms of θ.

1 mark

Sub. $(2, b)$ into tangent line equation:

$2\cos\theta + b\sin\theta = 2 + 2\cos\theta$

$b\sin\theta = 2$

$b = \dfrac{2}{\sin\theta}$

ii. Point D, with coordinates $(4, d)$, is on the line segment XY.

Find d in terms of θ.

1 mark

Sub. $(4, d)$:

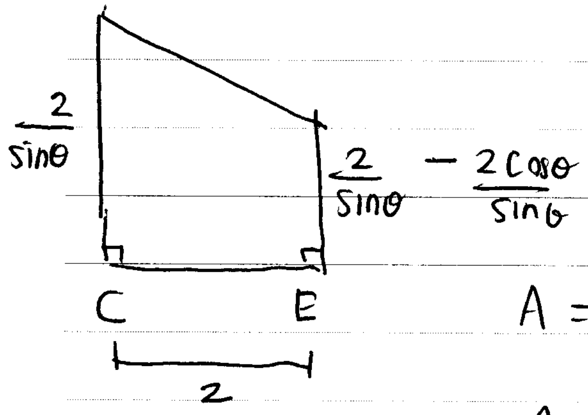
$4\cos\theta + d\sin\theta = 2 + 2\cos\theta$

$d\sin\theta = 2 - 2\cos\theta$

$d = \dfrac{2}{\sin\theta} - \dfrac{2\cos\theta}{\sin\theta}$

d. Consider the trapezium *CEDB* with parallel sides of length *b* and *d*.

Find the value of θ for which the area of the trapezium *CEDB* is a minimum. Also find the minimum value of the area. 3 marks



$$A = \frac{h}{2}(a+b)$$

$$\therefore A = \frac{2}{2}\left(\frac{2}{\sin\theta} + \frac{2}{\sin\theta} - \frac{2\cos\theta}{\sin\theta}\right)$$

$$\therefore A = \frac{4}{\sin\theta} - \frac{2\cos\theta}{\sin\theta}$$

$$A(\theta) = \frac{4-2\cos\theta}{\sin\theta}$$

Let $u = 4-2\cos\theta$, $u' = 2\sin\theta$
$v = \sin\theta$, $v' = \cos\theta$

$$A'(\theta) = \frac{vu'-uv'}{v^2}$$

$$= \frac{2\sin^2\theta - \cos\theta(4-2\cos\theta)}{\sin^2\theta}$$

$$A'(\theta) = \frac{2\sin^2\theta + 2\cos^2\theta - 4\cos\theta}{\sin^2\theta}$$

Use $\sin^2\theta + \cos^2\theta = 1$ (Pythagorean Identity)

$$\therefore A'(\theta) = \frac{2-4\cos\theta}{\sin^2\theta}$$

$A'(\theta) = 0$ when $2 - 4\cos\theta = 0$

$\therefore \cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$

$$A_{min} = A\left(\frac{\pi}{3}\right)$$

$$= \frac{4-2\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)}$$

$$= \frac{4-2\times\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{3}{\frac{\sqrt{3}}{2}} = \frac{6}{\sqrt{3}}$$

$$= 2\sqrt{3}$$

END OF QUESTION AND ANSWER BOOK

Minimum area $= 2\sqrt{3}$ sq units