SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

Let $f: R \to R$, $f(x) = 2\sin(3x) - 3$.

The period and range of this function are respectively

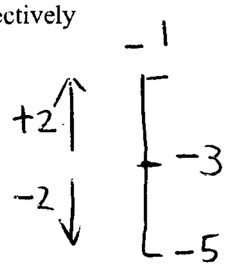
A. period = $\frac{2\pi}{3}$ and range = $[-5, -1]$

~~B.~~ period = $\frac{2\pi}{3}$ and range = $[-2, 2]$

~~C.~~ period = $\frac{\pi}{3}$ and range = $[-1, 5]$

~~D.~~ period = 3π and range = $[-1, 5]$

~~E.~~ period = 3π and range = $[-2, 2]$



Range: $[-5, -1]$
Only A has the correct range!

Question 2

The inverse function of $f: (-2, \infty) \to R$, $f(x) = \frac{1}{\sqrt{x+2}}$ is

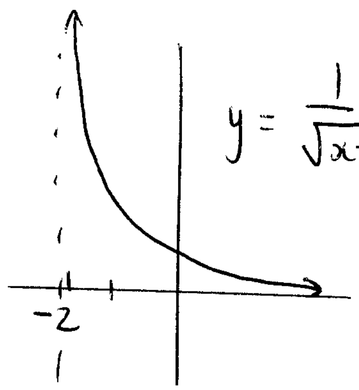
A. $f^{-1}: R^+ \to R$ $\quad f^{-1}(x) = \frac{1}{x^2} - 2$

~~B.~~ $f^{-1}: R\backslash\{0\} \to R$ $\quad f^{-1}(x) = \frac{1}{x^2} - 2$

C. $f^{-1}: R^+ \to R$ $\quad f^{-1}(x) = \frac{1}{x^2} + 2$

~~D.~~ $f^{-1}: (-2, \infty) \to R$ $\quad f^{-1}(x) = x^2 + 2$

~~E.~~ $f^{-1}: (2, \infty) \to R$ $\quad f^{-1}(x) = \frac{1}{x^2 - 2}$



$y = \frac{1}{\sqrt{x+2}}$

dom(f)	ran(f)
$(-2,\infty)$	$(0,\infty)$
dom(f^{-1})	ran(f^{-1})
$(0,\infty)$	$(-2,\infty)$

$y = \frac{1}{\sqrt{x+2}}$

$\downarrow$

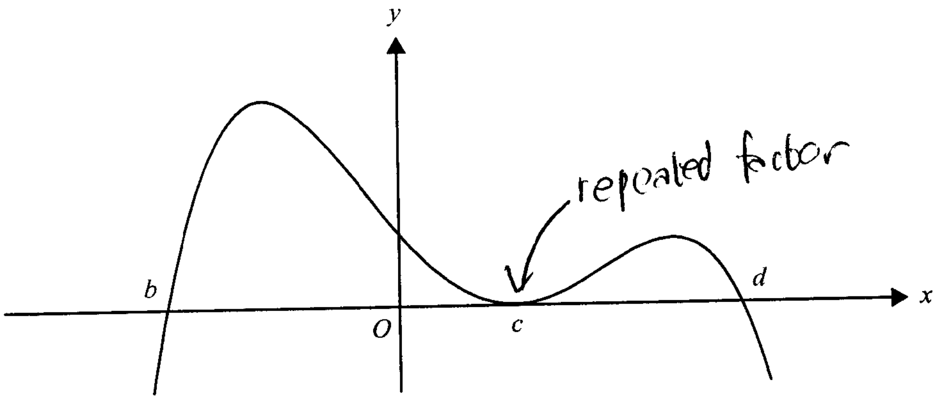
$x = \frac{1}{\sqrt{y+2}}$

$\frac{1}{x^2} = y+2$

$y = \frac{1}{x^2} - 2$

Only C, A have correct domain

Question 3



The rule for a function with the graph above could be

~~**A.**~~ $y = -2(x + b)(x - c)^2(x - d)$

~~**B.**~~ $y = 2(x + b)(x - c)^2(x - d)$

C. $y = -2(x - b)(x - c)^2(x - d)$

~~**D.**~~ $y = 2(x - b)(x - c)(x - d)$

~~**E.**~~ $y = -2(x - b)(x + c)^2(x + d)$

$y = -a(x-b)(x-c)^2(x-d)$

All factors have the form: $(x - x\text{-intercept value})$

Question 4

Consider the tangent to the graph of $y = x^2$ at the point (2, 4).

Which of the following points lies on this tangent?

A. (1, –4)

B. (3, 8)

C. (–2, 6)

D. (1, 8)

E. (4, –4)

$y = x^2$

$\frac{dy}{dx} = 2x$

$\left.\frac{dy}{dx}\right|_{x=2} = 4$

$\therefore$ Equation of tangent:

$m = 4, \ (x_1, y_1) = (2, 4)$

$\therefore y - 4 = 4(x - 2)$

$y - 4 = 4x - 8$

$y = 4x - 4$

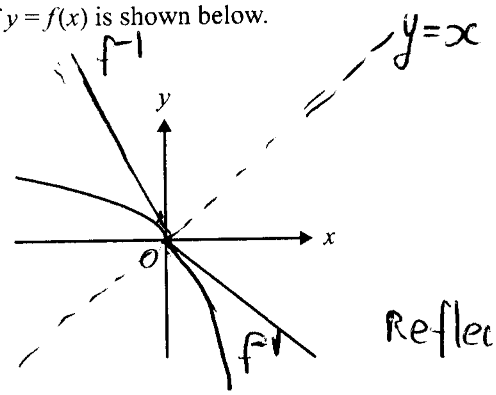
Test co-ordinates:

A. $-4 = 4 \times 1 - 4$ false

B. $8 = 4 \times 3 - 4$ true

Question 5

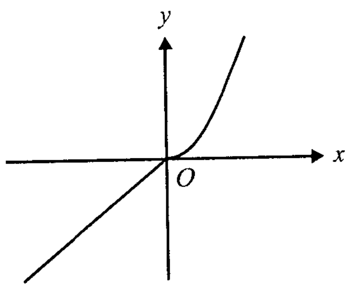
Part of the graph of $y = f(x)$ is shown below.



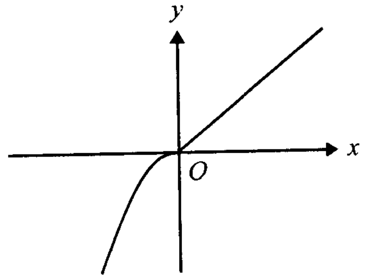
Reflect in line $y = x$

The corresponding part of the graph of the inverse function $y = f^{-1}(x)$ is best represented by

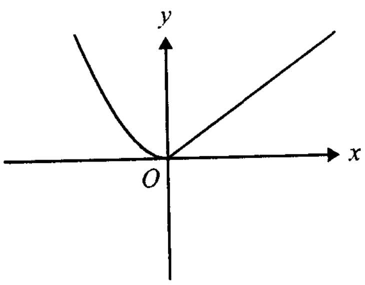
A.



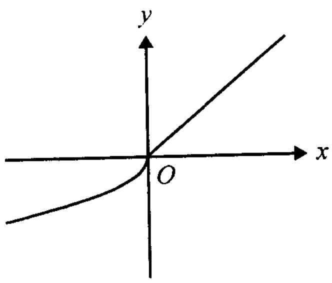
B.



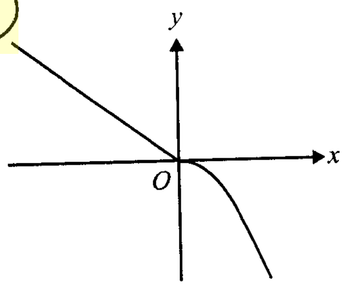
C.



D.



E.



Question 6

For the polynomial $P(x) = x^3 - ax^2 - 4x + 4$, $P(3) = 10$, the value of a is

A. -3
B. -1
C. 1
D. 3
E. 10

$10 = 3^3 - a \times 3^2 - 4 \times 3 + 4$

$10 = 27 - 9a - 12 + 4$

$10 = 19 - 9a$ $\therefore -9 = -9a$

$\therefore a = 1$

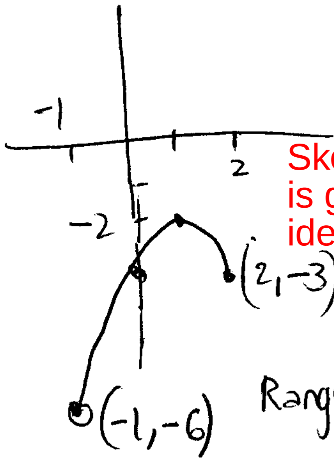
Question 7

The range of the function $f: (-1, 2] \to R$, $f(x) = -x^2 + 2x - 3$ is

A. R
B. $(-6, -3]$
C. $(-6, -2]$
D. $[-6, -3]$
E. $[-6, -2]$

$y = -x^2 + 2x - 3$ has a maximum tp when $x = 1$

$f(1) = -1 + 2 - 3 = -2$

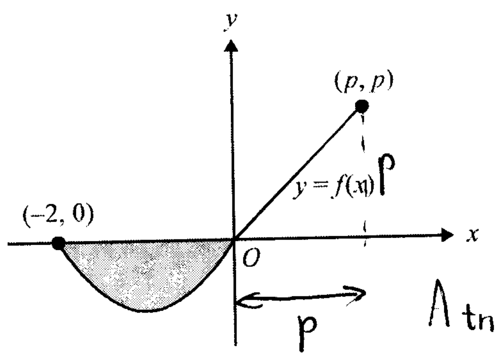


Sketch graph is good idea!

Range: $(-6, -2]$

Question 8

The graph of a function $f: [-2, p] \to R$ is shown below.



Δ triangle $= \frac{bh}{2}$

$= \frac{p^2}{2}$

The average value of f over the interval $[-2, p]$ is zero.

The area of the shaded region is $\frac{25}{8}$.

If the graph is a straight line, for $0 \le x \le p$, then the value of p is

A. 2
B. 5
C. $\frac{5}{4}$
D. $\frac{5}{2}$
E. $\frac{25}{4}$

$\int_0^p f(x)\,dx = \frac{25}{8}$

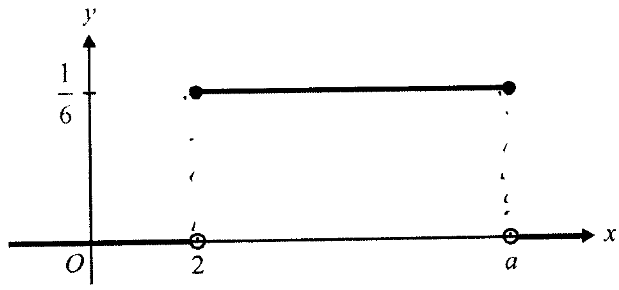
$\therefore \frac{p^2}{2} = \frac{25}{8}$

$p^2 = \frac{25}{4}$

$p = \sqrt{\frac{25}{4}} = \frac{5}{2}$

Question 9

The graph of the probability density function of a continuous random variable, X, is shown below.



$(a-2)\times\frac{1}{6}=1$

$\therefore a-2=6$

$\therefore a=8$

If $a > 2$, then $E(X)$ is equal to

A. 8

B. 5

C. 4

D. 3

E. 2

$$E(X)=\int_2^a \frac{1}{6}\cdot x\,dx$$
$$=\int_2^a \frac{x}{6}\,dx$$
$$=\int_2^8 \frac{x}{6}\,dx$$
$$=\left[\frac{x^2}{12}\right]_2^8$$
$$=\frac{64}{12}-\frac{4}{12}=\frac{60}{12}$$
$$=5$$

Question 10

The binomial random variable, X, has $E(X) = 2$ and $\mathrm{Var}(X)=\frac{4}{3}$.

$\Pr(X = 1)$ is equal to

A. $\left(\frac{1}{3}\right)^6$

B. $\left(\frac{2}{3}\right)^6$

C. $\frac{1}{3}\times\left(\frac{2}{3}\right)^2$

D. $6\times\frac{1}{3}\times\left(\frac{2}{3}\right)^5$

E. $6\times\frac{2}{3}\times\left(\frac{1}{3}\right)^5$

$np=2$

$np(1-p)=\frac{4}{3}$

$\therefore 2(1-p)=\frac{4}{3}$

$1-p=\frac{2}{3}$

$\therefore p=\frac{1}{3}$

and $n=6$.

$$\Pr(x=1)=\binom{6}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^5$$
$$=6\times\frac{1}{3}\times\left(\frac{2}{3}\right)^5$$

Question 11

The transformation that maps the graph of $y = \sqrt{8x^3 + 1}$ onto the graph of $y = \sqrt{x^3 + 1}$ is a

A. dilation by a factor of 2 from the y-axis.

B. dilation by a factor of 2 from the x-axis.

C. dilation by a factor of $\frac{1}{2}$ from the x-axis.

D. dilation by a factor of 8 from the y-axis.

E. dilation by a factor of $\frac{1}{2}$ from the y-axis.

$f(x) = \sqrt{8x^3+1}$

$y' = f\left(\frac{x}{2}\right) = \sqrt{8\left(\frac{x}{2}\right)^3+1} = \sqrt{x^3+1}$

$\therefore y' = f\left(\frac{x}{2}\right)$

which is a dilation of factor 2 away from y-axis

Question 12

A box contains five red balls and three blue balls. John selects three balls from the box, without replacing them.

The probability that at least one of the balls that John selected is red is

A. $\frac{5}{7}$

B. $\frac{5}{14}$

C. $\frac{7}{28}$

D. $\frac{15}{56}$

E. $\frac{55}{56}$

Let X = no. of reds

$Pr(X \geq 1) = 1 - Pr(X=0)$

$= 1 - Pr(3\text{ blue})$

$Pr(3\text{ blue})$

$\xrightarrow{\frac{3}{8}} B \xrightarrow{\frac{2}{7}} B \xrightarrow{\frac{1}{6}} B$

$= \frac{3}{8} \times \frac{2}{7} \times \frac{1}{6}$

$\therefore Pr(X \geq 1) = 1 - \frac{\cancel{3}^1}{\cancel{8}_4} \times \frac{\cancel{2}^1}{7} \times \frac{1}{\cancel{6}_2}$

$= 1 - \frac{1}{56}$

$= \frac{55}{56}$

Question 13

The function f is a probability density function with rule

$$f(x)=\begin{cases} ae^{x} & 0\le x\le 1 \\ ae & 1<x\le 2 \\ 0 & \text{otherwise}\end{cases}$$

The value of a is

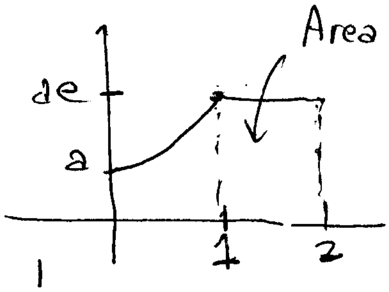
A. 1

B. e

C. $\frac{1}{e}$

D. $\frac{1}{2e}$

E. $\frac{1}{2e-1}$



Area $= ae \times 1$
$= ae$

$$1=\int_0^1 ae^{x}\,dx + ae$$
$$=\left[ae^{x}\right]_0^1 + ae$$
$$=(ae-ae^{0})+ae$$
$$=2ae-a$$
$$\therefore 2ae-a=1$$
$$\therefore a(2e-1)=1$$
$$a=\frac{1}{2e-1}$$

Question 14

Consider the following discrete probability distribution for the random variable X.

x	1	2	3	4	5
$\Pr(X=x)$	$\frac{1}{15}$ p	$2p$ $\frac{2}{15}$	$3p$ $\frac{3}{15}$	$4p$ $\frac{4}{15}$	$5p$ $\frac{5}{15}$

The mean of this distribution is

A. 2

B. 3

C. $\frac{7}{2}$

D. $\frac{11}{3}$

E. 4

$$p+2p+3p+4p+5p=1$$
$$15p=1$$
$$\therefore p=\frac{1}{15}$$

$$E(X)=1\times\frac{1}{15}+2\times\frac{2}{15}+3\times\frac{3}{15}+4\times\frac{4}{15}+5\times\frac{5}{15}$$
$$=\frac{1+4+9+16+25}{15}$$
$$=\frac{55}{15}=\frac{11}{3}$$

Question 15

If $\int_0^5 g(x)dx = 20$ and $\int_0^5 (2g(x) + ax)dx = 90$, then the value of a is

A. 0
B. 4
C. 2
D. −3
E. 1

$2\int_0^5 g(x)\,dx + \int_0^5 ax\,dx = 90$

$\therefore 2\times 20 + \int_0^5 ax\,dx = 90 \quad \therefore \left[\frac{ax^2}{2}\right]_0^5 = 50$

$\frac{25a}{2} = 50 \quad \therefore a = 4$

Question 16

Let $f(x) = ax^m$ and $g(x) = bx^n$, where a, b, m and n are positive integers. The domain of f = domain of $g = R$.

If $f'(x)$ is an antiderivative of $g(x)$, then which one of the following must be true?

A. $\frac{m}{n}$ is an integer

B. $\frac{n}{m}$ is an integer

C. $\frac{a}{b}$ is an integer

D. $\frac{b}{a}$ is an integer

E. $n - m = 2$

$f'(x) = max^{m-1}$

$\therefore max^{m-1} = \int bx^n\,dx$

or: $m(m-1)ax^{m-2} = bnx^n$

This requires: $m-n=2$ and: $\frac{b}{a}$ is an integer

since: $\frac{b}{a} = m(m-1)$

Question 17

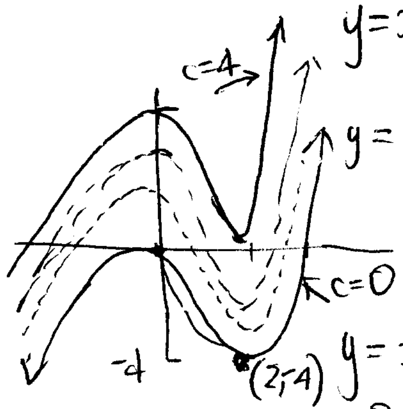
A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts.

The set of all possible values of c is

A. R
B. R^+
C. $\{0, 4\}$
D. $(0, 4)$
E. $(-\infty, 4)$

Q18 is not on the curriculum any more.

Consider: $y = x^3 - 3x^2$



$\frac{dy}{dx} = 3x^2 - 6x = 0$

$3x(x-2) = 0$

$\therefore x = 0, 2$

When $x=2$, $y=-4$

$y = x^3 - 3x^2 + c$ has 3 x-intercepts if $0 < c < 4$

Question 19

If $f(x) = \int_0^x \left(\sqrt{t^2+4}\right) dt$, then $f'(-2)$ is equal to

A. $\sqrt{2}$

B. $-\sqrt{2}$

C. $2\sqrt{2}$

D. $-2\sqrt{2}$

E. $4\sqrt{2}$

$f(x) = \int_0^x \left(\sqrt{t^2+4}\right) dt = \left[F(t)\right]_0^x$

$f(x) = F(x) - F(0)$

$\therefore f'(x) = F'(x) - 0$

$= \sqrt{x^2+4}$

$\therefore f'(-2) = \sqrt{8}$

$= 2\sqrt{2}$

Question 20

If $f(x-1) = x^2 - 2x + 3$, then $f(x)$ is equal to

A. $x^2 - 2$

B. $x^2 + 2$

C. $x^2 - 2x + 2$

D. $x^2 - 2x + 4$

E. $x^2 - 4x + 6$

Let $x - 1 = t$ $\therefore x = t+1$

$\therefore f(t) = (t+1)^2 - 2(t+1) + 3$

$f(t) = t^2 + 2t + 1 - 2t - 2 + 3$

$= t^2 + 2$

$\therefore f(t) = t^2 + 2$

$\therefore f(x) = x^2 + 2$

Question 21

The graphs of $y = mx + c$ and $y = ax^2$ will have no points of intersection for all values of m, c and a such that

A. $a > 0$ and $c > 0$

B. $a > 0$ and $c < 0$

C. $a > 0$ and $c > -\frac{m^2}{4a}$

D. $a < 0$ and $c > -\frac{m^2}{4a}$

E. $m > 0$ and $c > 0$

$y = mx + c$

$y = ax^2$

$ax^2 = mx + c$

$ax^2 - mx - c = 0$

For no intersection points, $\Delta < 0$

$\therefore (-m)^2 - 4(a)(-c) < 0$

$m^2 + 4ac < 0$

If $a < 0$

$4ac < -m^2$

$\therefore c > \frac{-m^2}{4a}$

$\therefore$ This matches D.

If $a > 0$: then:

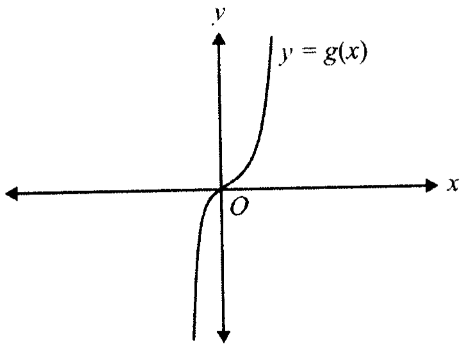
$4ac < -m^2$

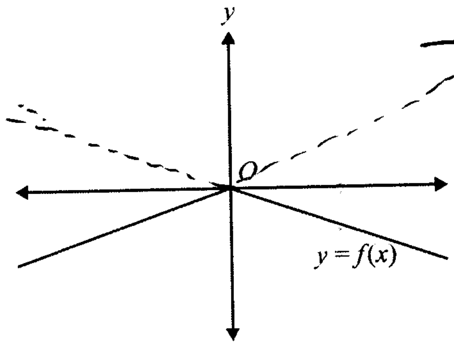
$c < \frac{-m^2}{4a}$ (since $a > 0$, dividing by a does not affect the inequality)

This does not match any option

Question 22

The graphs of the functions with rules $f(x)$ and $g(x)$ are shown below.

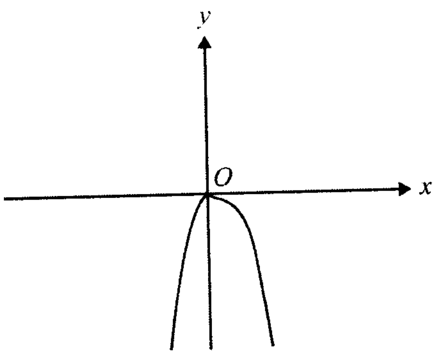




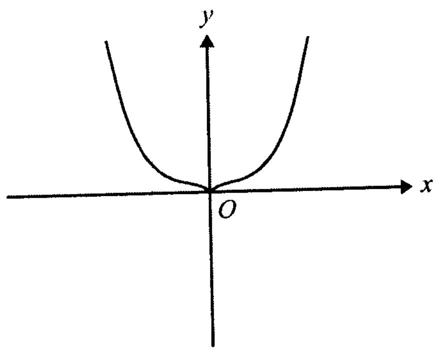
$-f(x)=h(x)$

Which one of the following best represents the graph of the function with rule $g(-f(x))$?

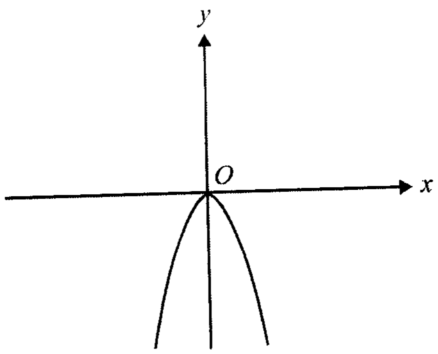
~~A.~~



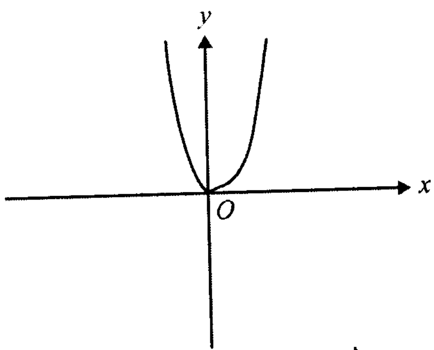
B.



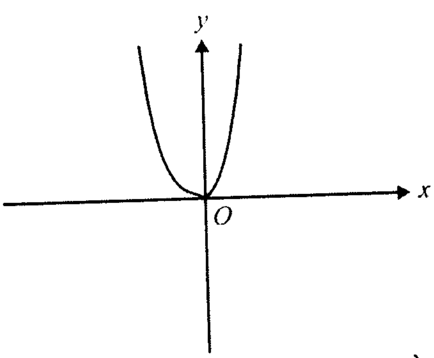
C.



~~D.~~



~~E.~~



From the symmetrical nature of the graph of $-f(x)=h(x)$

$h(p)=h(-p)$ ∴ (For example: $h(2)=h(-2)$)

∴ $g(h(x))=g(h(-x))$

∴ $g(h(x))$ must be symmetrical in y-axis.

END OF SECTION 1
TURN OVER

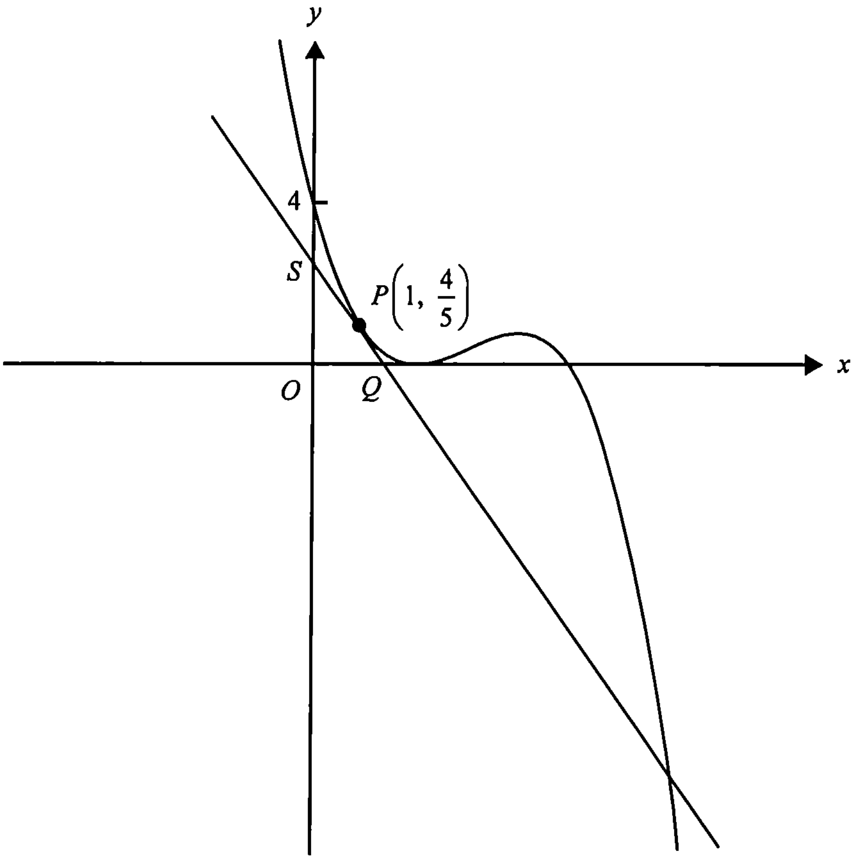
Eliminate:

A, D, E

Since $h(x)$ is always positive and $g(x)>0$ for $x>0$, $g(h(x))>0$ So B is correct

Question 1 (9 marks)

Let $f: R \to R$, $f(x) = \frac{1}{5}(x-2)^2(5-x)$. The point $P\left(1, \frac{4}{5}\right)$ is on the graph of f, as shown below.
The tangent at P cuts the y-axis at S and the x-axis at Q.



a. Write down the derivative $f'(x)$ of $f(x)$. 1 mark

$f'(x) = \frac{-3(x-4)(x-2)}{5}$

b. **i.** Find the equation of the tangent to the graph of f at the point $P\left(1, \frac{4}{5}\right)$. 1 mark

$f'(1) = \frac{-9}{5}$ $\therefore y - \frac{4}{5} = -\frac{9}{5}(x-1)$

$y - \frac{4}{5} = \frac{-9x}{5} + \frac{9}{5}$

$y = -\frac{9x}{5} + \frac{13}{5}$

ii. Find the coordinates of points Q and S. 2 marks

$S: \left(0, \frac{13}{5}\right)$

$Q: \left(\frac{13}{9}, 0\right)$

c. Find the distance PS and express it in the form $\frac{\sqrt{b}}{c}$, where b and c are positive integers. 2 marks

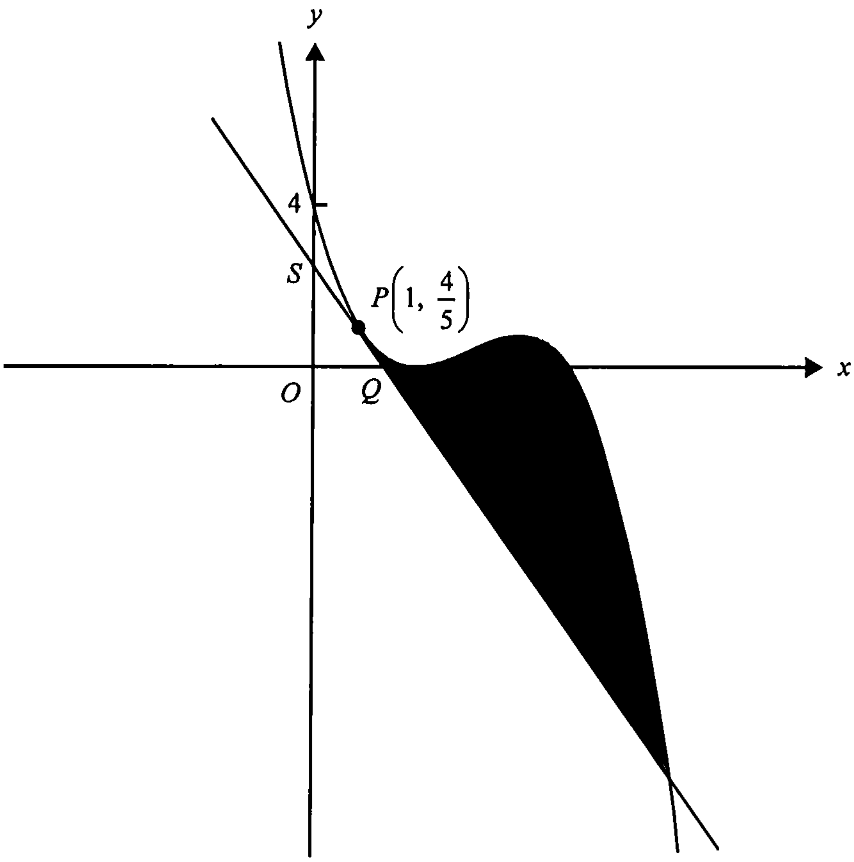
$P = \left(1, \frac{4}{5}\right)$, $S = \left(0, \frac{13}{5}\right)$

$$PS = \sqrt{(1-0)^2 + \left(\frac{4}{5} - \frac{13}{5}\right)^2}$$

$$= \sqrt{1 + \frac{81}{25}}$$

$$= \sqrt{\frac{106}{25}}$$

$$= \frac{\sqrt{106}}{5}$$



d. Find the area of the shaded region in the graph above. 3 marks

Intersection points:

$\frac{1}{5}(x-2)^2(5-x) = \frac{-9x}{5} + \frac{13}{5}$

Solving : $x = 1, 7$

Required area

$= \int_1^7 f(x) - \left(-\frac{9x}{5} + \frac{13}{5}\right) dx$

$= \int_1^7 \left(f(x) + \frac{9x}{5} - \frac{13}{5}\right) dx$

$= \frac{108}{5}$ sq. units

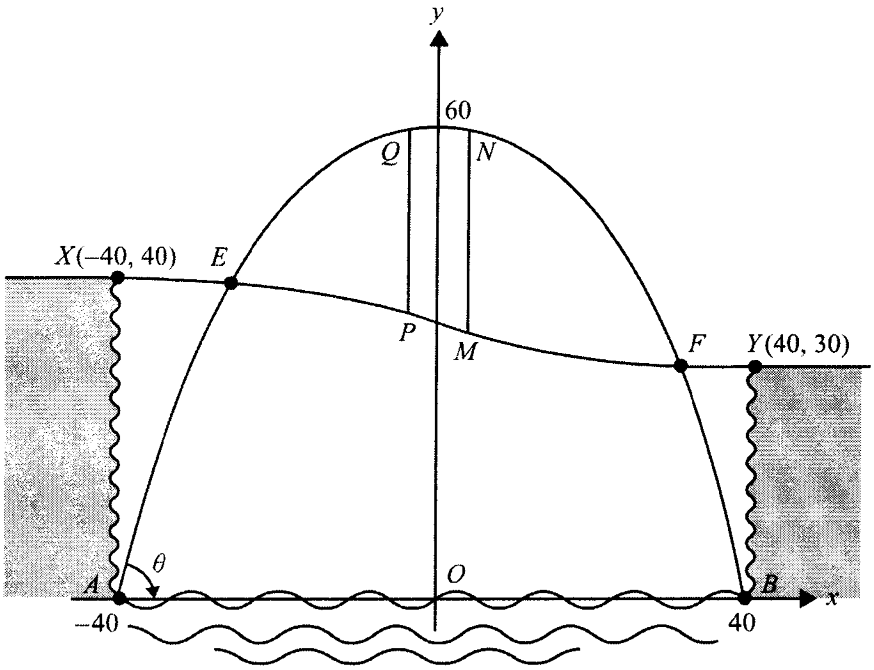
Question 2 (14 marks)

A city is located on a river that runs through a gorge.

The gorge is 80 m across, 40 m high on one side and 30 m high on the other side.

A bridge is to be built that crosses the river and the gorge.

A diagram for the design of the bridge is shown below.



The main frame of the bridge has the shape of a parabola. The parabolic frame is modelled by $y = 60 - \frac{3}{80}x^2$ and is connected to concrete pads at $B(40, 0)$ and $A(-40, 0)$.

The road across the gorge is modelled by a cubic polynomial function.

a. Find the angle, θ, between the tangent to the parabolic frame and the horizontal at the point $A(-40, 0)$ to the nearest degree. 2 marks

$$y = 60 - \frac{3x^2}{80}$$

$$\frac{dy}{dx} = -\frac{6x}{80}$$

$$\text{At } x = -40, \left.\frac{dy}{dx}\right|_{x=-40} = \frac{-6}{80} \times -40 = 3$$

$$\tan(\theta) = 3$$

$$\theta = \tan^{-1}(3)$$

$$\theta \approx 72^\circ$$

The road from X to Y across the gorge has gradient zero at $X(-40, 40)$ and at $Y(40, 30)$, and has equation $y = \frac{x^3}{25600} - \frac{3x}{16} + 35$.

b. Find the maximum downwards slope of the road. Give your answer in the form $-\frac{m}{n}$ where m and n are positive integers. 2 marks

$$\frac{dy}{dx} = \frac{3x^2}{256000} - \frac{3}{16}$$

Greatest downward slope occus when $x = 0$

$$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{3}{16}$$

Two vertical supporting columns, MN and PQ, connect the road with the parabolic frame.
The supporting column, MN, is at the point where the vertical distance between the road and the parabolic frame is a maximum.

c. Find the coordinates (u, v) of the point M, stating your answers correct to two decimal places. 3 marks

Let $f(x) = 60 - \frac{3x^2}{80}$ and $g(x) = \frac{x^3}{25600} - \frac{3x}{16} + 35$

Then, the vertical separation between the road and the frame is $f(x) - g(x)$.

Let $h(x) = f(x) - g(x) = 60 - \frac{3x^2}{80} - \frac{x^3}{25600} + \frac{3x}{16} - 35$

At M, $h'(x) = 0$. Solving: $x = 2.4903099$ (reject $x = -642.4$ as $x > 0$)

$g(2.4903099) = 34.53367$

$M = (2.49, 34.53)$

The second supporting column, PQ, has its lowest point at $P(-u, w)$.

d. Find, correct to two decimal places, the value of w and the lengths of the supporting columns MN and PQ. 3 marks

At P, $x = -2.4903099$

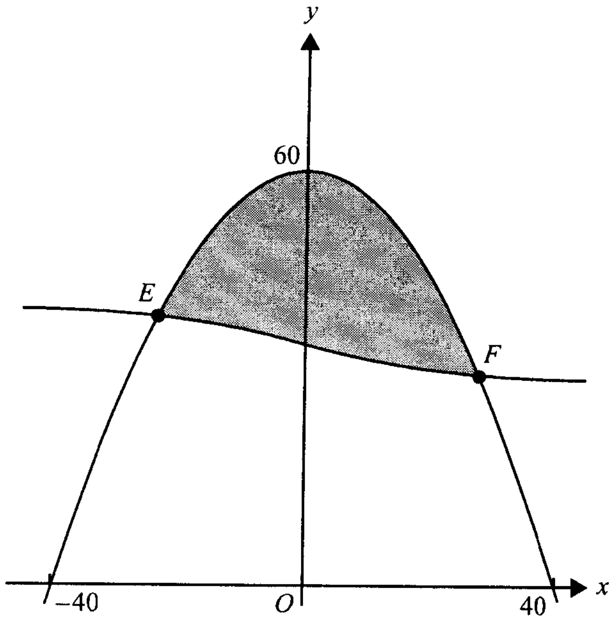
and $y = g(-2.4903099)$
$= 35.46633$

$\therefore w = 35.47$

Length of $\overline{NM} = h(2.4903099) \approx 25.23$ m

Length of $\overline{PQ} = h(-2.4903099)$
$= 24.30$ m

For the opening of the bridge, a banner is erected on the bridge, as shown by the shaded region in the diagram below.



e. Find the x-coordinates, correct to two decimal places, of E and F, the points at which the road meets the parabolic frame of the bridge. 3 marks

At E and F, $60 - \frac{3x^2}{80} = \frac{x^3}{25600} - \frac{3x}{16} + 35$

$x = -23.71, 28.00$

f. Find the area of the banner (shaded region), giving your answer to the nearest square metre. 1 mark

Required area $= \int_{-23.71}^{28.00} \left(60 - \frac{3x^2}{80} - \frac{x^3}{25600} + \frac{3x}{16} - 35\right) dx$

$\approx 870 \text{ m}^2$

Question 3 (11 marks)

Mani is a fruit grower. After his oranges have been picked, they are sorted by a machine, according to size. Oranges classified as **medium** are sold to fruit shops and the remainder are made into orange juice.

The distribution of the diameter, in centimetres, of medium oranges is modelled by a continuous random variable, X, with probability density function

$$f(x)=\begin{cases}\frac{3}{4}(x-6)^2(8-x) & 6\le x\le 8\\ 0 & \text{otherwise}\end{cases}$$

a. **i.** Find the probability that a randomly selected medium orange has a diameter greater than 7 cm. 2 marks

$$\Pr(X>7)=\int_7^8 \frac{3}{4}(x-6)^2(8-x)\,dx$$
$$=\frac{11}{16}$$

ii. Mani randomly selects three medium oranges.

Find the probability that exactly one of the oranges has a diameter greater than 7 cm.

Express the answer in the form $\frac{a}{b}$, where a and b are positive integers. 2 marks

Let Y = no. of oranges with diameter >7cm

$Y \stackrel{d}{=} Bi\left(n=3, p=\frac{11}{16}\right)$

$\Pr(Y=1)=\binom{3}{1}\left(\frac{11}{16}\right)^1\left(\frac{5}{16}\right)^2=\frac{825}{4096}$

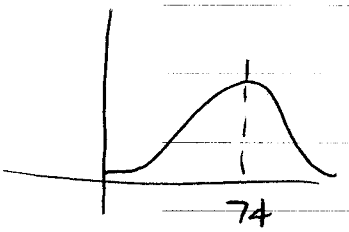
b. Find the mean diameter of medium oranges, in centimetres. 1 mark

$$E(X)=\int_6^8 x\,f(x)\,dx$$
$$=\int_6^8 \frac{3x}{4}(x-6)^2(8-x)\,dx$$
$$=\frac{36}{5}$$

For oranges classified as **large**, the quantity of juice obtained from each orange is a normally distributed random variable with a mean of 74 mL and a standard deviation of 9 mL.

c. What is the probability, correct to three decimal places, that a randomly selected large orange produces less than 85 mL of juice, given that it produces more than 74 mL of juice? 2 marks

$L \overset{d}{=} N(\mu = 74, \sigma = 9)$



$\Pr(L<85 \mid L>74) = \dfrac{\Pr(74<L<85)}{\Pr(L>74)}$

$= \dfrac{\Pr(74<L<85)}{0.5} = \dfrac{0.389188}{0.5}$

$= 0.778$

Mani also grows lemons, which are sold to a food factory. When a truckload of lemons arrives at the food factory, the manager randomly selects and weighs four lemons from the load. If one or more of these lemons is underweight, the load is rejected. Otherwise it is accepted.

It is known that 3% of Mani's lemons are underweight.

d. **i.** Find the probability that a particular load of lemons will be rejected. Express the answer correct to four decimal places. 2 marks

$\Pr(\text{rejected}) = \Pr(X \geq 1)$ $\quad$ X = no. of underweight lemons

$X \overset{d}{=} Bi\left(n=4, p=\frac{3}{100}\right)$

$= 1 - \Pr(X=0) = 0.1147$

ii. Suppose that instead of selecting only four lemons, n lemons are selected at random from a particular load.

Find the smallest integer value of n such that the probability of at least one lemon being underweight exceeds 0.5 2 marks

$1 - \Pr(X=0) \geq 0.5$

$\therefore 0.5 \geq \Pr(X=0)$

First solve:

$\binom{n}{0}(0.03)^0(0.97)^n = 0.5$

$\therefore (0.97)^n = 0.5$

$n = 22.8$

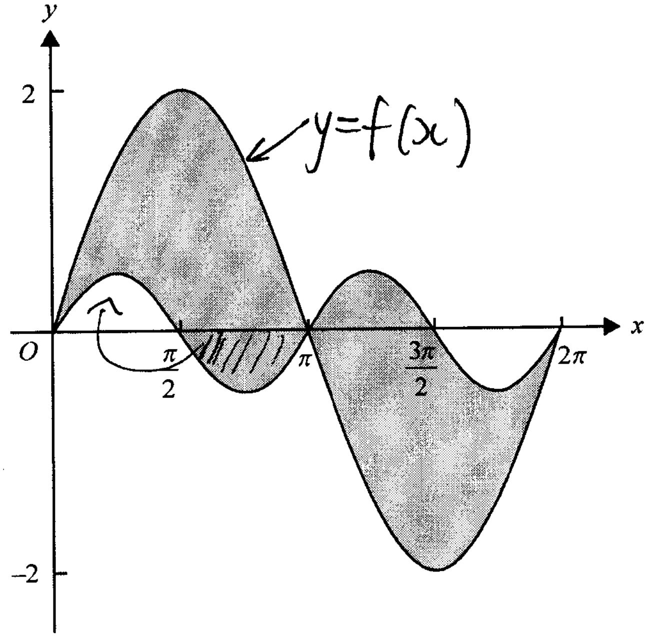
$\therefore n = 23$

Question 4 (9 marks)

An electronics company is designing a new logo, based initially on the graphs of the functions

$f(x) = 2\sin(x)$ and $g(x) = \frac{1}{2}\sin(2x)$, for $0 \le x \le 2\pi$.

These graphs are shown in the diagram below, in which the measurements in the x and y directions are in metres.



The logo is to be painted onto a large sign, with the area enclosed by the graphs of the two functions (shaded in the diagram) to be painted red.

a. The total area of the shaded regions, in square metres, can be calculated as $a\int_0^{\pi} \sin(x)dx$.

What is the value of a? 1 mark

$$\text{Total area} = 2\int_0^{\pi} f(x)\,dx$$

$$= 2\int_0^{\pi} 2\sin x\,dx$$

$$= 4\int_0^{\pi} \sin x\,dx$$

$\therefore a = 4.$

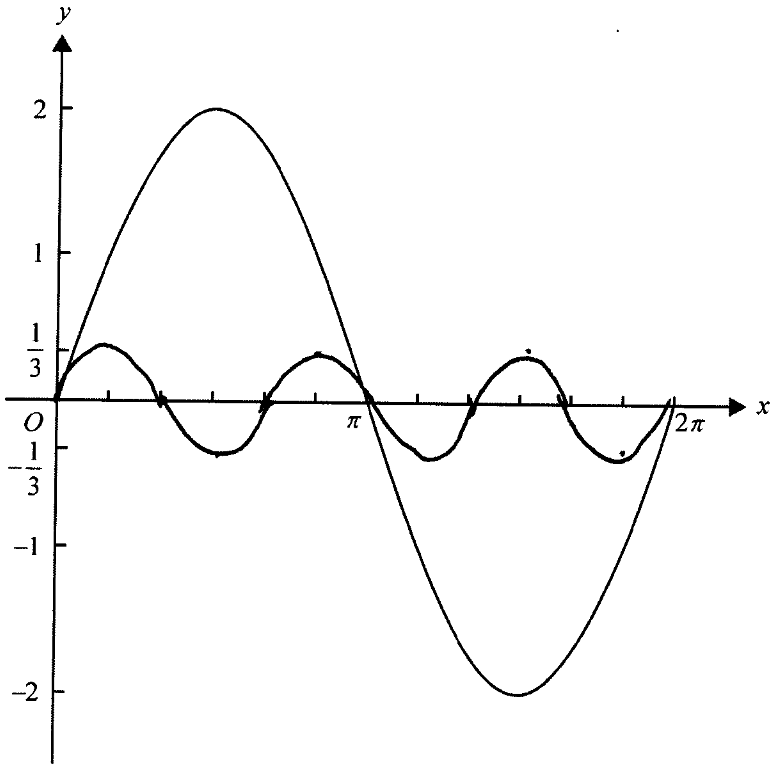
The electronics company considers changing the circular functions used in the design of the logo.

Its next attempt uses the graphs of the functions $f(x) = 2\sin(x)$ and $h(x) = \frac{1}{3}\sin(3x)$, for $0 \le x \le 2\pi$.

b. On the axes below, the graph of $y = f(x)$ has been drawn.

On the same axes, draw the graph of $y = h(x)$.

2 marks



c. State a sequence of two transformations that maps the graph of $y = f(x)$ to the graph of $y = h(x)$.

2 marks

$f(x) = 2\sin x$

$f_2(x) = \frac{1}{6}f(x) = \frac{1}{3}\sin x$

$f_3(x) = f_2(3x) = \frac{1}{3}\sin(3x)$

- Dilation by factor $\frac{1}{6}$ away from x-axis
- Dilation by factor $\frac{1}{3}$ away from y-axis.

The electronics company now considers using the graphs of the functions $k(x) = m\sin(x)$ and $q(x) = \frac{1}{n}\sin(nx)$, where m and n are positive integers with $m \geq 2$ and $0 \leq x \leq 2\pi$.

d. **i.** Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if n is even.**

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers. 2 marks

Area $= 2\int_0^{\pi} k(x) - q(x)\,dx$

$= 2\int_0^{\pi} m\sin x - \frac{1}{n}\sin nx\,dx = 2\left(2m + \frac{\cos(n\pi)-1}{n^2}\right)$

If n is even, $\cos(n\pi) = 1$ so Area $= 4m$

ii. Find the area enclosed by the graphs of $y = k(x)$ and $y = q(x)$ in terms of m and n **if n is odd.**

Give your answer in the form $am + \frac{b}{n^2}$, where a and b are integers. 2 marks

If n is odd, $\cos(n\pi) = -1$

so

Area $= 2\left(2m + \frac{-1-1}{n^2}\right)$

$= 2\left(2m - \frac{2}{n^2}\right)$

$= 4m - \frac{4}{n^2}$

Question 5 (15 marks)

a. Let $S(t) = 2e^{\frac{t}{3}} + 8e^{\frac{-2t}{3}}$, where $0 \le t \le 5$.

i. Find $S(0)$ and $S(5)$. 1 mark

$S(0) = 2e^{0} + 8e^{0} = 10$

$S(5) = 2e^{5/3} + 8e^{\frac{-10}{3}}$

ii. The minimum value of S occurs when $t = \log_e(c)$.

State the value of c and the minimum value of S. 2 marks

$\frac{dS}{dt} = \frac{2}{3}e^{t/3} - \frac{16}{3}e^{-\frac{2t}{3}}$

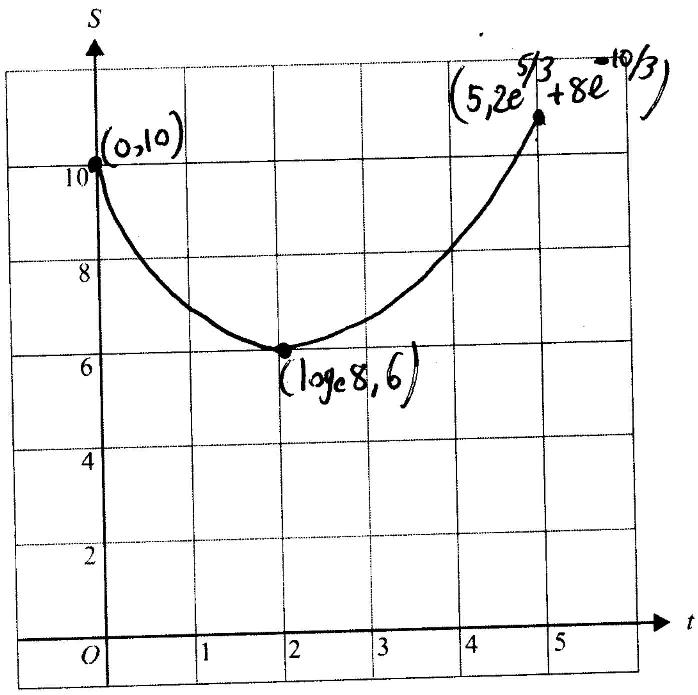
Let $\frac{dS}{dt} = 0$ $\therefore \frac{2}{3}e^{t/3} = \frac{16}{3}e^{-2t/3}$

$\therefore e^{t/3} = \frac{8}{e^{2t/3}}$ $\therefore e^{t} = 8$ $\therefore t = \log_e 8$

$c = 8$, $S(\log_e 8) = 6$

$\therefore S_{min} = 6$

iii. On the axes below, sketch the graph of S against t for $0 \le t \le 5$. Label the end points and the minimum point with their coordinates. 2 marks



iv. Find the value of the average rate of change of the function S over the interval $[0, \log_e(c)]$. 2 marks

$$\frac{S(\log_e 8) - S(0)}{\log_e 8 - 0} = \frac{6-10}{\log_e 8}$$

$$= \frac{-4}{\log_e 8}$$

Let $V : [0, 5] \rightarrow R$, $V(t) = de^{\frac{t}{3}} + (10 - d)e^{-\frac{2t}{3}}$, where d is a real number and $d \in (0, 10)$.

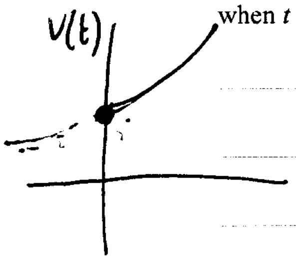
b. If the minimum value of the function occurs when $t = \log_e(9)$, find the value of d. 2 marks

For a minimum, let $V'(t) = 0$.

$\therefore t = \log_e\left(-2\left(\frac{d-10}{d}\right)\right)$ where $0 < d < 10$

$\therefore \frac{-2(d-10)}{d} = 9$ $\therefore -2d + 20 = 9d$ $\therefore d = \frac{20}{11}$

c. **i.** Find the set of possible values of d such that the minimum value of the function occurs when $t = 0$. 2 marks

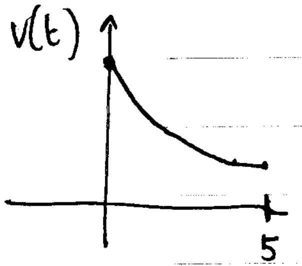


We require: $V'(0) \geq 0$ for endpoint minimum at $t = 0$

$\therefore \frac{3d-20}{3} > 0$ $\therefore \frac{20}{3} \leq d < 10$

$\therefore d \geq \frac{20}{3}$

ii. Find the set of possible values of d such that the minimum value of the function occurs when $t = 5$. 2 marks



We require: $V'(5) \leq 0$

$\therefore \frac{\left(d\left(e^5+2\right)-20\right)e^{\frac{-10}{3}}}{3} \leq 0$

$\therefore d(e^5+2) - 20 \leq 0$

$\therefore d \leq \frac{20}{e^5+2}$

But $d > 0$

$\therefore 0 < d \leq \frac{20}{e^5+2}$