Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Product Rule

a. Differentiate x^3e^{2x} with respect to x.

$y = x^3 e^{2x}$

$u = x^3 \qquad v = e^{2x}$

$\frac{dy}{dx} = vu' + uv'$

$u' = 3x^2 \qquad v' = 2e^{2x}$

$\frac{dy}{dx} = 3x^2e^{2x} + 2x^3e^{2x}$

2 marks

b. For $f(x) = \log_e(x^2 + 1)$, find $f'(2)$.

$f'(x) = \frac{2x}{x^2+1}$

$f'(2) = \frac{2 \times 2}{2^2+1}$

2 marks

$= \frac{4}{5}$

TURN OVER

Question 2

a. Find an antiderivative of $\cos(2x + 1)$ with respect to x.

$$\int \cos(2x+1)\,dx = \frac{1}{2}\sin(2x+1) + c, \; c \in R$$

1 mark

This question requires knowledge of the modulus, which is no longer on the course.

Question 3

Let $f: R^+ \to R$ where $f(x) = \frac{1}{x^2}$.

a. Find $g(x) = f(f(x))$.

$$f(f(x)) = \frac{1}{[f(x)]^2}$$

$$g(x) = \frac{1}{\left(\frac{1}{x^2}\right)^2}$$

$$g(x) = x^4$$

1 mark

b. Evaluate $g^{-1}(16)$, where g^{-1} is the inverse function of g.

~~$\text{dom}(f(f(x))) = \text{dom}(f(x)) = R^+$~~

$\text{dom}(g)$	$\text{ran}(g)$
R^+	R^+
$\text{dom}(g^{-1})$	$\text{ran}(g^{-1})$
R^+	R^+

Let $g^{-1}(16) = p$.

Then: $(16, p)$ lies on graph of g^{-1} $\therefore$ $(p, 16)$ lies on graph of g

$\therefore p^4 = 16, \quad p > 0$

$\therefore p = 2 \qquad \therefore g^{-1}(16) = 2$

1 mark

Question 4

a. Write down the amplitude and period of the function

$$f: R \rightarrow R, f(x) = 4\sin\left(\frac{x+\pi}{3}\right).$$

Amplitude $= 4$

Period $= \dfrac{2\pi}{\frac{1}{3}} = 6\pi$

2 marks

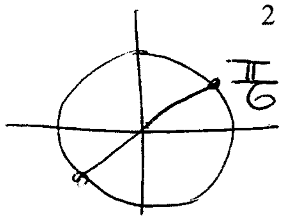
b. Solve the equation $\sqrt{3}\sin(x) = \cos(x)$ for $x \in [-\pi, \pi]$.

$\sqrt{3}\sin x = \cos x, \quad -\pi \leq x \leq \pi$

$\therefore \tan x = \dfrac{1}{\sqrt{3}}$

$\therefore x = \dfrac{\pi}{6}, \dfrac{-5\pi}{6}$

2 marks



Question 5 is on the next page.

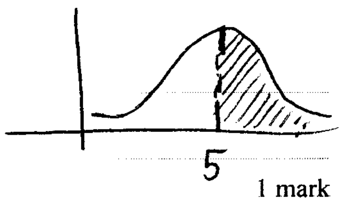
TURN OVER

Question 5

Let X be a normally distributed random variable with mean 5 and variance 9 and let Z be the random variable with the standard normal distribution.

a. Find $\Pr(X > 5)$.

$\Pr(x > 5) = 0.5$



1 mark

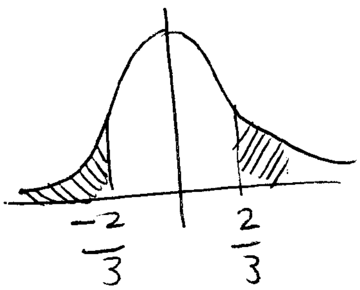
b. Find b such that $\Pr(X > 7) = \Pr(Z < b)$.

$\sigma^2 = 9 \quad \therefore \sigma = \sqrt{9} = 3$

$\Pr(x > 7) = \Pr\left(z > \frac{7}{3}\right)$

$= \Pr\left(z > \frac{2}{3}\right)$

2 marks



$= \Pr\left(z < -\frac{2}{3}\right)$

$\therefore b = -\frac{2}{3}$

TURN OVER

Question 6

The transformation $T: R^2 \rightarrow R^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

The image of the curve $y = 2x^2 + 1$ under the transformation T has equation $y = ax^2 + bx + c$.
Find the values of a, b and c.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3x - 1 \\ 2y + 4 \end{bmatrix}$$

$x' = 3x - 1 \quad \therefore \quad \frac{x'+1}{3} = x$

$y' = 2y + 4 \quad \therefore \quad \frac{y'-4}{2} = y$

$y = 2x^2 + 1$

$\therefore \frac{y'-4}{2} = 2\left(\frac{x'+1}{3}\right)^2 + 1$

$\frac{y'-4}{2} = \frac{2(x'+1)^2}{9} + 1$

3 marks

$y' - 4 = \frac{4(x'+1)^2}{9} + 2$

$9y' - 36 = 4(x'+1)^2 + 18$

Dropping primes:

$9y - 36 = 4(x^2 + 2x + 1) + 18$

$9y = 4x^2 + 8x + 4 + 18 + 36$

$9y = 4x^2 + 8x + 58$

$y = \frac{4x^2}{9} + \frac{8x}{9} + \frac{58}{9}$

$a = \frac{4}{9}, \quad b = \frac{8}{9}, \quad c = \frac{58}{9}$

Question 7

The continuous random variable X has a distribution with probability density function given by

$$f(x) = \begin{cases} ax(5-x) & \text{if } 0 \le x \le 5 \\ 0 & \text{if } x < 0 \text{ or if } x > 5 \end{cases}$$

where a is a positive constant.

a. Find the value of a.

$$\int_0^5 ax(5-x)\,dx = 1$$

$$\therefore a\int_0^5 (5x - x^2)\,dx = 1$$

$$\therefore a\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5 = 1$$

$$a\left(\left(\frac{5^3}{2} - \frac{5^3}{3}\right) - 0\right) = 1$$

$$\therefore 125a\left(\frac{1}{2} - \frac{1}{3}\right) = 1$$

$$\frac{125a}{6} = 1$$

$$a = \frac{6}{125}$$

3 marks

b. Express $\Pr(X < 3)$ as a definite integral. (Do **not** evaluate the definite integral.)

$$\Pr(X < 3) = \int_0^3 \frac{6}{125}x(5-x)\,dx$$

1 mark

Question 8

The discrete random variable X has the probability distribution

x	-1	0	1	2
$\Pr(X = x)$	p^2	p^2	$\frac{p}{4}$	$\frac{4p+1}{8}$

Find the value of p.

$$p^2 + p^2 + \frac{p}{4} + \frac{4p+1}{8} = 1$$

$$2p^2 + \frac{p}{4} + \frac{4p+1}{8} = 1$$

$$\therefore 16p^2 + 2p + 4p + 1 = 8$$

$$16p^2 + 6p - 7 = 0$$

$$(2p-1)(8p+7) = 0$$

$$p = \frac{1}{2} \text{ or } p = \frac{-7}{8}$$

But $0 < p < 1$

$$\therefore p = \frac{1}{2}$$

3 marks

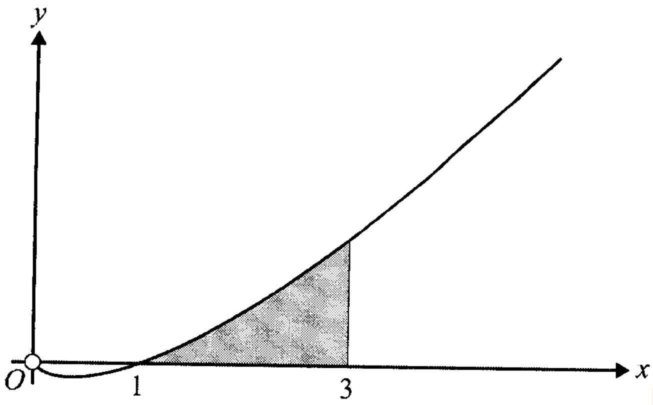
$2p \quad -1$

$8p \quad +7$

$14p - 8p = 6p$

Question 9

Part of the graph of $f: R^+ \rightarrow R, f(x) = x \log_e(x)$ is shown below.



Product Rule

$u = x^2 \qquad v = \log_e x$

$u' = 2x \qquad v' = \frac{1}{x}$

a. Find the derivative of $x^2 \log_e(x)$.

$\frac{dy}{dx} = 2x\log_e x + x^2 \cdot \frac{1}{x}$

$= 2x\log_e x + x$

1 mark

b. Use your answer to **part a.** to find the area of the shaded region in the form $a \log_e(b) + c$ where a, b and c are non-zero real constants.

We need to evaluate:

$\int_1^3 x\log_e x \, dx$

From above:

$\frac{d}{dx}\left(x^2\log_e x\right) = 2x\log_e x + x$

$\therefore x^2\log_e x = 2\int x\log_e x \, dx + \int x \, dx$

$x^2\log_e x = 2\int x\log_e x \, dx + \frac{x^2}{2}$

$x^2\log_e x - \frac{x^2}{2} = 2\int x\log_e x \, dx$

$\frac{x^2\log_e x}{2} - \frac{x^2}{4} = \int x\log_e x \, dx$

3 marks

$\int_1^3 x\log_e x \, dx = \left[\frac{x^2\log_e x}{2} - \frac{x^2}{4}\right]_1^3$

$= \left(\frac{9}{2}\log_e 3 - \frac{9}{4}\right) - \left(\frac{1^2\log_e 1}{2} - \frac{1}{4}\right)$

$= \frac{9}{2}\log_e 3 - 2$

Question 10

The line $y = ax - 1$ is a tangent to the curve $y = x^{\frac{1}{2}} + d$ at the point $(9, c)$ where a, c and d are real constants. Find the values of a, c and d.

$\left.\frac{dy}{dx}\right|_{x=9} = a$ $\qquad y = x^{\frac{1}{2}} + d$. $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\therefore \frac{1}{2\sqrt{9}} = a \quad \therefore a = \frac{1}{2\times 3} = \frac{1}{6}$

Equation of tangent: $y = \frac{1}{6}x - 1$ Since $(9, c)$ lies on tangent, $\therefore c = \frac{9}{6} - 1 \quad \therefore c = \frac{1}{2}$

$\therefore$ Point $(9, \frac{1}{2})$ lies on $y = x^{1/2} + d$

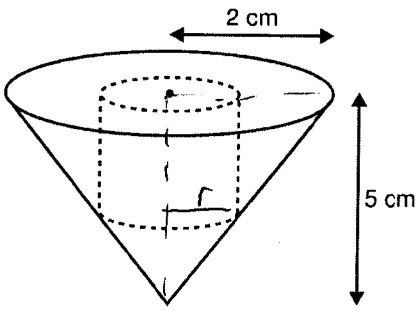
$\therefore \frac{1}{2} = \sqrt{9} + d \quad \therefore d = \frac{1}{2} - 3 = \frac{-5}{2}$

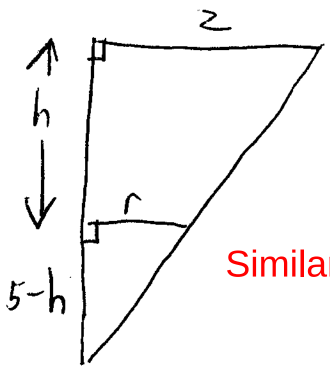
$\therefore a = \frac{1}{6}, \quad c = \frac{1}{2}, \quad d = -\frac{5}{2}$

4 marks

Question 11

A cylinder fits exactly in a right circular cone so that the base of the cone and one end of the cylinder are in the same plane as shown in the diagram below. The height of the cone is 5 cm and the radius of the cone is 2 cm. The radius of the cylinder is r cm and the height of the cylinder is h cm.





Similar triangles

For the cylinder inscribed in the cone as shown above

a. find h in terms of r

$\frac{r}{2} = \frac{5-h}{5}$

$\frac{5r}{2} = 5 - h$

$\therefore h = 5 - \frac{5r}{2}$

2 marks

Question 11 – continued

The total surface area, S cm^2, of a cylinder of height h cm and radius r cm is given by the formula

$$S = 2\pi rh + 2\pi r^2.$$

b. find S in terms of r

$$S = 2\pi rh + 2\pi r^2$$

$$S = 2\pi r\left(5 - \frac{5r}{2}\right) + 2\pi r^2$$

$$S = 10\pi r - 5\pi r^2 + 2\pi r^2$$

$$S = 10\pi r - 3\pi r^2$$

1 mark

c. find the value of r for which S is a maximum.

$$S'(r) = 10\pi - 6\pi r$$

For a maximum, $S'(r) = 0$

$$\therefore 10\pi - 6\pi r = 0$$

$$r = \frac{10}{6} = \frac{5}{3}$$

2 marks

END OF QUESTION AND ANSWER BOOK