Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

a. Differentiate $\sqrt{4-x}$ with respect to x.

$$y = \sqrt{4-x}$$

$$\therefore y = (4-x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \times (4-x)^{-\frac{1}{2}} \times -1$$

$$= -\frac{1}{2\sqrt{4-x}}$$

1 mark

b. If $g(x) = x^2\sin(2x)$, find $g'\left(\frac{\pi}{6}\right)$.

Let $u = x^2$ $\quad V = \sin(2x)$

$u' = 2x$ $\quad V' = 2\cos(2x)$

$$g'(x) = 2x\sin(2x) + 2x^2\cos(2x)$$

$$g'\left(\frac{\pi}{6}\right) = 2 \times \frac{\pi}{6}\sin\left(\frac{\pi}{3}\right) + 2\left(\frac{\pi}{6}\right)^2 \times \cos\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} \times \frac{\sqrt{3}}{2} + 2 \times \frac{1}{2} \times \frac{\pi^2}{36}$$

$$= \frac{\sqrt{3}\pi}{6} + \frac{\pi^2}{36}$$

2 marks

Question 2

a. Find an antiderivative of $\frac{1}{3x-4}$ with respect to x.

$$\int \frac{1}{3x-4}\,dx = \frac{1}{3}\log_e(3x-4) + c$$

Note: "+c" is optional, as question asked for an antiderivative.

1 mark

b. Solve the equation $4^x - 15 \times 2^x = 16$ for x.

$$4^x - 15 \times 2^x = 16$$

Let $2^x = a$

$$4^x = (2^2)^x = (2^x)^2 = a^2$$

$$\therefore a^2 - 15a = 16$$

$$a^2 - 15a - 16 = 0$$

$$(a-16)(a+1) = 0$$

$$a = 16, -1$$

But $a = 2^x$

$\therefore 2^x = 16$ or $2^x = -1$

$2^x = 2^4$; no real solution

$\therefore x = 4$

3 marks

Question 3

a. State the range and period of the function

$$h: R \to R,\ h(x) = 4 + 3\cos\left(\frac{\pi x}{2}\right).$$

7 $\uparrow +3$

4

1 $\downarrow -3$

Range: $[1, 7]$

Period: $\dfrac{2\pi}{\frac{\pi}{2}} = 4$

2 marks

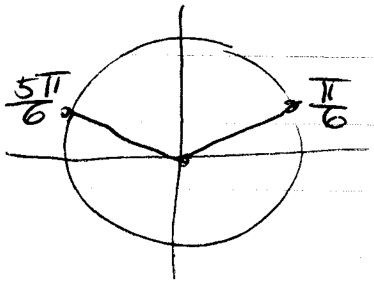
b. Solve the equation

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \text{ for } x \in [0, \pi].$$

$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}, \quad 0 \le x \le \pi$

$\therefore 0 \le 2x \le 2\pi$

$\therefore \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le \frac{7\pi}{3}$



$2x + \frac{\pi}{3} = \frac{5\pi}{6}, \ 2\pi + \frac{\pi}{6}$

$\therefore 2x + \frac{\pi}{3} = \frac{5\pi}{6}, \ \frac{13\pi}{6}$

$\therefore 2x = \frac{3\pi}{6}, \ \frac{11\pi}{6}$

$2x = \frac{\pi}{2}, \ \frac{11\pi}{6}$

$\therefore x = \frac{\pi}{4}, \ \frac{11\pi}{12}$

2 marks

Question 4

If the function f has the rule $f(x)=\sqrt{x^2-9}$ and the function g has the rule $g(x)=x+5$

a. find integers c and d such that $f(g(x)) = \sqrt{(x+c)(x+d)}$

$$f(g(x)) = \sqrt{(g(x))^2-9}$$
$$= \sqrt{(x+5)^2-9}$$
$$= \sqrt{(x+5)^2-3^2}$$
$$= \sqrt{(x+5-3)(x+5+3)}$$
$$= \sqrt{(x+2)(x+8)}$$

2 marks

$\therefore c=2, d=8$
(or $c=8, d=2$)

b. state the maximal domain for which $f(g(x))$ is defined.

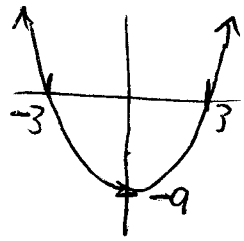
2 marks

$f(g(x))$ is defined
if $\text{ran}(g) \subseteq \text{dom}(f)$

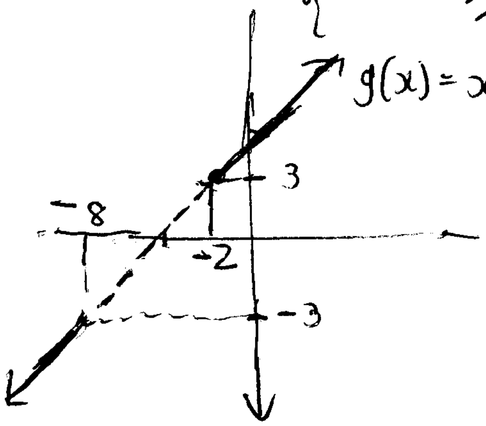
$f(x)=\sqrt{x^2-9}$ has domain:

$\{x : x \geq 3 \cup x : x \leq -3\}$

$x^2-9 \geq 0$



Where parabola is above (or on) x axis:
$x \geq 3 \cup x \leq -3$



$\text{ran}(g) \subseteq \text{dom}(f)$
for:
$x \geq -2 \cup x \leq -8$

$\therefore$ Maximal domain of $f(g(x))$ is:
$\{x : x \geq -2\} \cup \{x : x \leq -8\}$

Question 5

The probability distribution function for the continuous random variable X is given by

$$f(x) = \begin{cases} |3-x| & \text{if} \quad 2 \le x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find $\Pr(X < 3.5)$.

This question is no longer appropriate as modulus functions are not on the course.

2 marks

b. Find $\Pr(X < 2.5 \mid X < 3.5)$.

2 marks

TURN OVER

Question 6

Consider the simultaneous linear equations

$$kx - 3y = k + 3$$
$$4x + (k + 7)y = 1$$

where k is a real constant.

$A \cdot X = K$

a. Find the value of k for which there are infinitely many solutions.

$$\begin{bmatrix} k & -3 \\ 4 & k+7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k+3 \\ 1 \end{bmatrix}$$

For no unique solution, $\det(A) = 0$

$\therefore k(k+7) - -3 \times 4 = 0$

$k^2 + 7k + 12 = 0$

$(k+4)(k+3) = 0 \quad \therefore k = -4, -3$

If $k = -4$:

$-4x - 3y = -1$
$4x + 3y = 1$

Same equation
$\therefore$ Infinitely many solutions

If $k = -3$

$-3x - 3y = 0$
$4x + 4y = 1$

Parallel lines
$\therefore$ No solution

$\therefore k = -4$ for infinitely many solutions

3 marks

b. Find the values of k for which there is a unique solution.

b. $k \in \mathbb{R} \setminus \{-4, -3\}$

1 mark

Question 7

A biased coin is tossed three times. The probability of a head from a toss of this coin is p.

a. Find, in terms of p, the probability of obtaining

i. three heads from the three tosses

$\xrightarrow{p} H \xrightarrow{p} H \xrightarrow{p} H$

p^3

ii. two heads and a tail from the three tosses.

Let X = no. of heads $X \stackrel{d}{=} Bi(n=3, p)$

$Pr(X=2) = \binom{3}{2} p^2 (1-p)$

$= 3p^2(1-p)$

1 + 1 = 2 marks

b. If the probability of obtaining three heads equals the probability of obtaining two heads and a tail, find p.

$3p^2(1-p) = p^3$

$p^3 - 3p^2(1-p) = 0$

$\therefore p^2(p - 3(1-p)) = 0$

$p^2(p + 3p - 3) = 0$

$p^2(4p - 3) = 0$

$p = 0$ or $p = \frac{3}{4}$

2 marks

Question 8

Two events, A and B, are such that $\Pr(A) = \frac{3}{5}$ and $\Pr(B) = \frac{1}{4}$.

If A' denotes the complement of A, calculate $\Pr(A' \cap B)$ when

a. $\Pr(A \cup B) = \frac{3}{4}$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\therefore \frac{3}{4} = \frac{3}{5} + \frac{1}{4} - \Pr(A \cap B)$$

$$\therefore \frac{1}{2} = \frac{3}{5} + \Pr(A \cap B)$$

$$\Pr(A \cap B) = \frac{1}{10}$$

	B	B'	
A	$\frac{1}{10}$		
A'			
	$\frac{1}{4}$		

$$\Pr(A' \cap B) = \frac{1}{4} - \frac{1}{10} = \frac{3}{20}$$

2 marks

b. A and B are mutually exclusive.

If A and B are mutually exclusive, $\Pr(A \cap B) = 0$.

	B	B'	
A	0		
A'	$1/4$		
	$\frac{1}{4}$		

$$\Pr(A' \cap B) = \frac{1}{4}$$

1 mark

Question 9

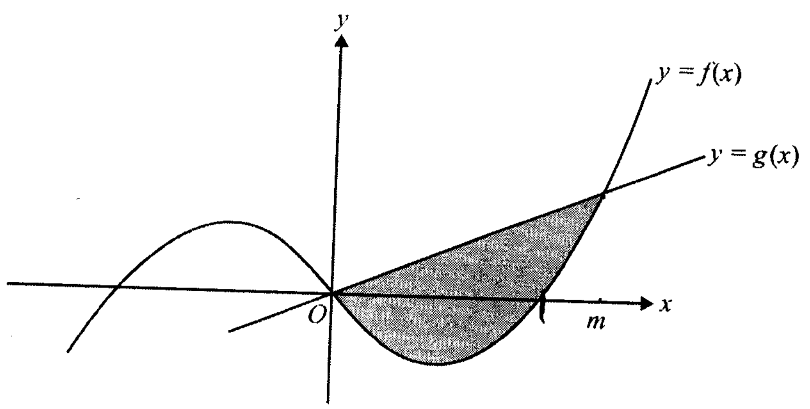
Parts of the graphs of the functions

$f: R \to R, f(x) = x^3 - ax \qquad a > 0$

$g: R \to R, g(x) = ax \qquad a > 0$

are shown in the diagram below.

The graphs intersect when $x = 0$ and when $x = m$.



Question 9 – continued

The area of the shaded region is 64.
Find the value of a and the value of m.

At the intersection point, $x = m$

$$m^3 - am = am$$
$$m^3 - 2am = 0$$
$$m(m^2 - 2a) = 0$$
$$m = \sqrt{2a} \quad (\text{since } m > 0)$$

$$\int_0^{\sqrt{2a}} (g(x) - f(x))\, dx = 64$$

$$\therefore \int_0^{\sqrt{2a}} ax - (x^3 - ax)\, dx = 64$$

$$\int_0^{\sqrt{2a}} (2ax - x^3)\, dx = 64$$

$$\therefore \left[ax^2 - \frac{x^4}{4}\right]_0^{\sqrt{2a}} = 64$$

$$a(\sqrt{2a})^2 - \frac{(\sqrt{2a})^4}{4} = 64$$

$$\therefore a \times 2a - \frac{4a^2}{4} = 64$$

$$2a^2 - a^2 = 64$$

$$a^2 = 64$$

$$\therefore a = \sqrt{64} = 8$$
$$(\text{since } a > 0)$$

and $m = \sqrt{2a} = \sqrt{16}$
$= 4$

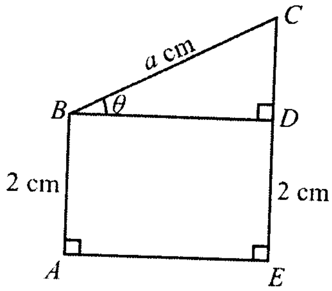
4 marks

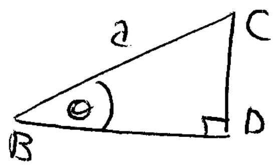
Question 10

The figure shown represents a wire frame where $ABCE$ is a convex quadrilateral. The point D is on line segment EC with $AB = ED = 2$ cm and $BC = a$ cm, where a is a positive constant.

$$\angle BAE = \angle CEA = \frac{\pi}{2}$$

Let $\angle CBD = \theta$ where $0 < \theta < \frac{\pi}{2}$.





a. Find BD and CD in terms of a and θ.

$\cos(\theta) = \frac{\overline{BD}}{a} \quad \therefore \quad \overline{BD} = a\cos\theta$

$\sin(\theta) = \frac{\overline{CD}}{a} \quad \therefore \quad \overline{CD} = a\sin\theta$

2 marks

b. Find the length, L cm, of the wire in the frame, including length BD, in terms of a and θ.

$L = a + a\sin\theta + a\cos\theta + 4 + a\cos\theta$

$L = a + a\sin\theta + 2a\cos\theta + 4$

1 mark

c. Find $\frac{dL}{d\theta}$, and **hence** show that $\frac{dL}{d\theta} = 0$ when $BD = 2CD$.

$$\frac{dL}{d\theta} = a\cos\theta - 2a\sin\theta$$

$= 0$ when

$a\cos\theta - 2a\sin\theta = 0$

$\therefore a\cos\theta = 2a\sin\theta$

$\therefore \overline{BD} = 2\,\overline{CD}$.

2 marks

d. Find the maximum value of L if $a = 3\sqrt{5}$.

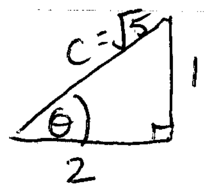
When $\frac{dL}{d\theta} = 0$

$\Rightarrow a\cos\theta = 2a\sin\theta$

$\therefore \cos\theta = 2\sin\theta$

$\therefore \frac{1}{2} = \tan\theta.$

$c^2 = 1^2 + 2^2 = 5$



$\therefore \sin\theta = \frac{1}{\sqrt{5}}$

$\cos\theta = \frac{2}{\sqrt{5}}$

$\therefore L = a + a\sin\theta + 2a\cos\theta + 4$

When $\tan\theta = \frac{1}{2}$ and $a = 3\sqrt{5}$,

1 mark

$L_{max} = 3\sqrt{5} + 3\sqrt{5} \times \frac{1}{\sqrt{5}} + 2 \times 3\sqrt{5} \times \frac{2}{\sqrt{5}} + 4$

$= 3\sqrt{5} + 3 + 12 + 4$

$= 19 + 3\sqrt{5}$

END OF QUESTION AND ANSWER BOOK