Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

a. If $y = (x^2 - 5x)^4$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 4(x^2-5x)^3 \times (2x-5)$$

$$= 4(2x-5)(x^2-5x)^3$$

1 mark

b. If $f(x) = \frac{x}{\sin(x)}$, find $f'\left(\frac{\pi}{2}\right)$.

$$y = \frac{x}{\sin x} \qquad \text{Let } u = x \quad u' = 1$$

$$V = \sin x \quad v' = \cos x$$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x}$$

2 marks

$$f'\left(\frac{\pi}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)}{\left(\sin\left(\frac{\pi}{2}\right)\right)^2}$$

$$= \frac{1-0}{1^2}$$

$$= 1$$

Question 2

Find an anti-derivative of $\dfrac{1}{(2x-1)^3}$ with respect to x.

NOTE: It was not essential to have the +c for full marks, since only **AN** antiderivative was requested

$$\int (2x-1)^{-3}\,dx$$

$$= \frac{(2x-1)^{-2}}{2x-2} + c$$

$$= \frac{-1}{4}(2x-1)^{-2} + c$$

$$= \frac{-1}{4(2x-1)^2} + c, \quad c \in R$$

Question 3

The rule for function h is $h(x) = 2x^3 + 1$. Find the rule for the inverse function h^{-1}.

dom(h)	ran(h)
R	R
dom(h^{-1})	ran(h^{-1})
R	R

$$y = 2x^3 + 1$$

$$x = 2y^3 + 1$$

2 marks

$$\frac{x-1}{2} = y^3$$

$$y = \sqrt[3]{\frac{x-1}{2}}$$

$$h^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$$

Question 4

On any given day, the number X of telephone calls that Daniel receives is a random variable with probability distribution given by

x	0	1	2	3
$\Pr(X = x)$	0.2	0.2	0.5	0.1

a. Find the mean of X.

$$E(X) = 0\times0.2 + 1\times0.2 + 2\times0.5 + 3\times0.1$$
$$= 0.2 + 1 + 0.3$$
$$= 1.5$$

2 marks

b. What is the probability that Daniel receives only one telephone call on each of three consecutive days?

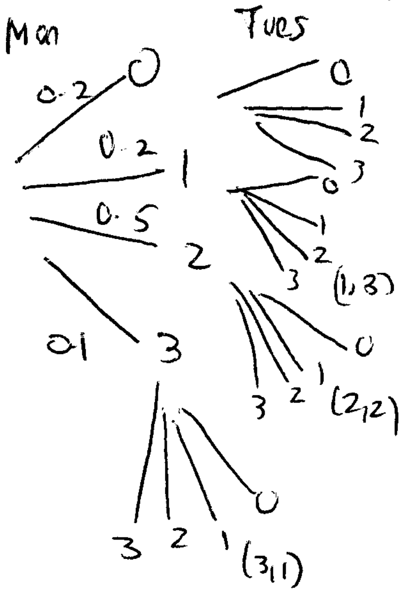
$$(0.2)^3 = \frac{1}{5}\times\frac{1}{5}\times\frac{1}{5}$$
$$= \frac{1}{125} \quad (\text{or } 0.008)$$

1 mark

c. Daniel receives telephone calls on both Monday and Tuesday.
What is the probability that Daniel receives a total of four calls over these two days?

$$\Pr(\text{Total} = 4 \text{ calls} \mid X \neq 0 \text{ for Monday and Tuesday})$$
$$= \frac{\Pr(\text{Total} = 4 \text{ calls} \cap X = 0 \text{ for Monday and Tuesday})}{\Pr(X \neq 0 \text{ for Monday and Tuesday})}$$

3 marks

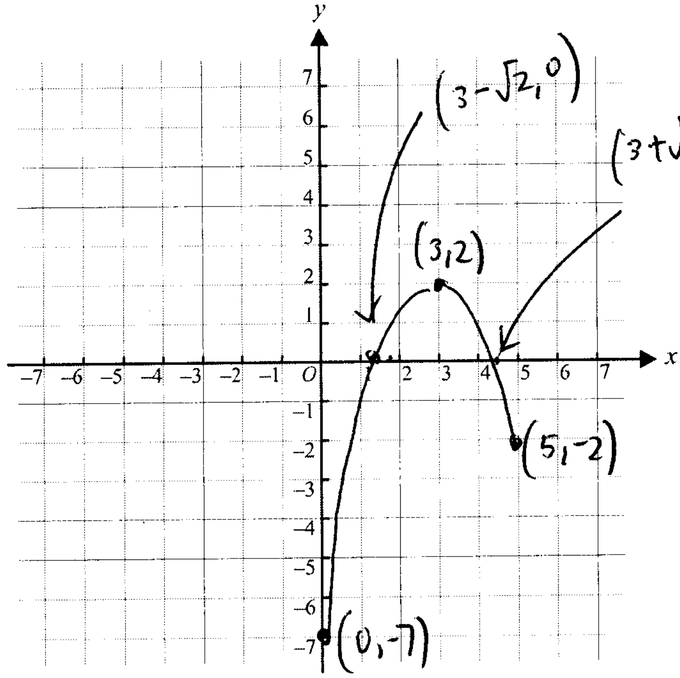


$$= \frac{\Pr(1,3) + \Pr(2,2) + \Pr(3,1)}{(0.8)^2}$$
$$= \frac{0.2\times0.1 + 0.5\times0.5 + 0.1\times0.2}{0.64}$$
$$= \frac{0.29}{0.64} = \frac{29}{64}$$

Question 5

a. Sketch the graph of f: $[0, 5] \rightarrow R$, $f(x) = -(x-3)^2+2$ Label the axes intercepts and endpoints with their coordinates.

$f(0) = -(-3)^2+2$
$= -9+2$
$= -7$
$f(5) = -(5-3)^2+2$
$= -4+2$
$= -2$



Let $y = 0$
$0 = -(x-3)^2+2$
$(x-3)^2 = 2$
$x-3 = \pm\sqrt{2}$
$x = 3 \pm \sqrt{2}$

3 marks

b. **i.** Find the coordinates of the image of the point (3, 2) under a reflection in the x-axis, followed by a translation of 5 units in the positive direction of the x-axis.

$(3,2) \rightarrow (3,-2) \rightarrow (8,-2)$

New point: $(8,-2)$

ii. Find the equation of the image of the graph of f under a reflection in the x-axis, followed by a translation of 5 units in the positive direction of the x-axis.

$f_1(x) = -(x-3)^2+2$
$f_2(x) = -f_1(x) = (x-3)^2-2$
$f_3(x) = f_2(x-5) = (x-5-3)^2-2$
$= (x-8)^2-2$

New equation: $y = (x-8)^2 - 2$

1 + 2 = 3 marks

Question 6

The graphs of $y = \cos(x)$ and $y = a\sin(x)$, where a is a real constant, have a point of intersection at $x = \frac{\pi}{3}$.

a. Find the value of a.

$\cos x = a \sin x$

If $x = \frac{\pi}{3}$, $\cos\left(\frac{\pi}{3}\right) = a \sin\left(\frac{\pi}{3}\right)$

$\therefore \frac{1}{2} = a \frac{\sqrt{3}}{2}$

$a = \frac{1}{\sqrt{3}}$

2 marks

b. If $x \in [0, 2\pi]$, find the x-coordinate of the other point of intersection of the two graphs.

$\frac{1}{\sqrt{3}} \sin x = \cos x$

$\tan x = \sqrt{3}$, $0 \leq x \leq 2\pi$

$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$ $\therefore$ Other intersection at $x = \frac{4\pi}{3}$

1 mark

Question 7

Solve the equation $2\log_e(x+2) - \log_e(x) = \log_e(2x+1)$, where $x > 0$, for x.

$2\log_e(x+2) - \log_e x = \log_e(2x+1)$

$\therefore \log_e(x+2)^2 - \log_e x = \log_e(2x+1)$

$\therefore \log_e\left(\frac{(x+2)^2}{x}\right) = \log_e(2x+1)$

$\therefore \frac{(x+2)^2}{x} = 2x+1$

3 marks

$(x+2)^2 = 2x^2 + x$

$x^2 + 4x + 4 = 2x^2 + x$

$\therefore 0 = x^2 - 3x - 4$

$(x-4)(x+1) = 0$

NOTE: for full marks you must show that the solution x = -1 is rejected

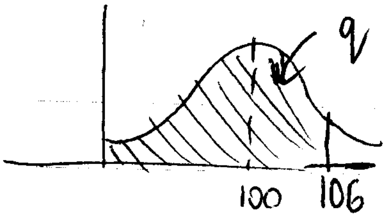
$x = 4, -1$

But $x > 0$, so $x = -1$ must be rejected.

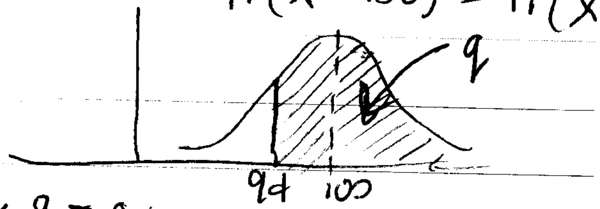
$\therefore x = 4$

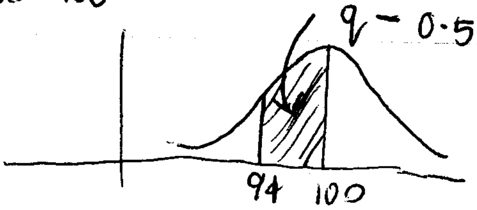
Question 8

a. The random variable X is normally distributed with mean 100 and standard deviation 4.
If $\Pr(X < 106) = q$, find $\Pr(94 < X < 100)$ in terms of q.



$\Pr(x < 106) = \Pr(x > 94)$





$\Pr(94 < x < 100) = q - 0.5$

2 marks

b. The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} \dfrac{x+1}{12} & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of b such that $\Pr(X \le b) = \dfrac{5}{8}$.

$$\int_0^b \frac{x+1}{12}\, dx = \frac{5}{8}$$

$$\therefore \frac{1}{12}\int_0^b (x+1)\, dx = \frac{5}{8}$$

$$\therefore \int_0^b (x+1)\, dx = \frac{5}{8} \times 12$$

$$\left[\frac{x^2}{2} + x\right]_0^b = \frac{15}{2}$$

$$\frac{b^2}{2} + b = \frac{15}{2}$$

$$b^2 + 2b - 15 = 0$$

$$(b-3)(b+5) = 0$$

$$\therefore b = 3, -5$$

3 marks

Since $b \in [0, 4]$, $b = 3$.

Question 9

a. Let $f: R \to R$, $f(x) = x\sin(x)$.
Find $f'(x)$.

Let $y = uv$ $\quad u = x \quad v = \sin x$
$u' = 1 \quad v' = \cos x$

$f'(x) = vu' + uv'$

$\therefore f'(x) = \sin x + x\cos x$

1 mark

b. Use the result of **part a.** to find the value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\cos(x)\,dx$ in the form $a\pi + b$.

You should have recognized that this is antidifferentiation by recognition

$$\frac{d}{dx}(x\sin x) = \sin x + x\cos x$$

$$\therefore x\sin x = \int \sin x\,dx + \int x\cos x\,dx$$

$$\therefore x\sin x = -\cos x + \int x\cos x\,dx$$

$$x\sin x + \cos x = \int x\cos x\,dx$$

$$\therefore \int_{\pi/6}^{\pi/2} x\cos x\,dx = \Big[x\sin x + \cos x\Big]_{\pi/6}^{\pi/2}$$

$$\int_{\pi/6}^{\pi/2} x\cos x\,dx = \left[\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right] - \left[\frac{\pi}{6}\sin\frac{\pi}{6} + \cos\frac{\pi}{6}\right]$$

$$= \frac{\pi}{2}\times 1 + 0 - \left(\frac{\pi}{6}\times\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$

3 marks

$$= \frac{\pi}{2} - \frac{\pi}{12} - \frac{\sqrt{3}}{2}$$

$$= \frac{5\pi}{12} - \frac{\sqrt{3}}{2}$$

Question 10

Let $f: R \to R$, $f(x) = e^{-mx} + 3x$, where m is a positive rational number.

a. **i.** Find, in terms of m, the x-coordinate of the stationary point of the graph of $y = f(x)$.

$$f'(x) = -me^{-mx} + 3$$

Let $f'(x) = 0$

$$-me^{-mx} = -3$$

$$e^{-mx} = \frac{3}{m}$$

$$-mx = \log_e\left(\frac{3}{m}\right)$$

$$x = \frac{1}{m}\log_e\left(\frac{m}{3}\right)$$

ii. State the values of m such that the x-coordinate of this stationary point is a positive number.

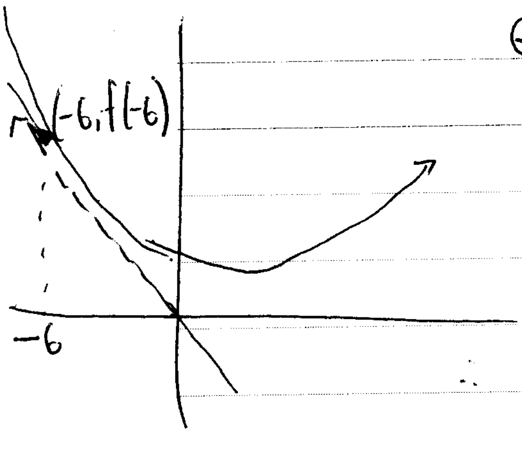
$m > 0 \quad \therefore \frac{1}{m}\log_e\left(\frac{m}{3}\right) > 0$ if $\log_e\left(\frac{m}{3}\right) > 0$

$\therefore \frac{m}{3} > 1$

$\therefore m > 3$

2 + 1 = 3 marks

b. For a particular value of m, the tangent to the graph of $y = f(x)$ at $x = -6$ passes through the origin.
Find this value of m.



Gradient of tangent

$= f'(-6)$

$= -me^{6m} + 3$

But we can also use: $\frac{y_2 - y_1}{x_2 - x_1}$

$\therefore \frac{f(-6) - 0}{-6 - 0} = 3 - me^{6m}$

$\therefore f(-6) = -6(3 - me^{6m})$

$f(-6) = -18 + 6me^{6m}$

3 marks

$\therefore e^{6m} - 18 = -18 + 6me^{6m}$

$\therefore e^{6m} = 6me^{6m}$

$\therefore 1 = 6m \quad \therefore m = \frac{1}{6}$

END OF QUESTION AND ANSWER BOOK