SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The function with rule $f(x) = 4\tan\left(\frac{x}{3}\right)$ has period

A. $\frac{\pi}{3}$
B. 6π
C. 3
D. 3π
E. $\frac{2\pi}{3}$

$$\frac{\pi}{\frac{1}{3}} = 3\pi$$

Question 2

For $f(x) = x^3 + 2x$, the average rate of change with respect to x for the interval $[1, 5]$ is

A. 18
B. 20.5
C. 24
D. 32.5
E. 33

$$\frac{f(5)-f(1)}{5-1} = \frac{135-3}{4} = \frac{132}{4} = 33$$

Question 3

This question involves a modulus function so is no longer on the syllabus

Question 4

If $f(x) = \frac{1}{2}e^{3x}$ and $g(x) = \log_e(2x) + 3$ then $g(f(x))$ is equal to

A. $2x^3 + 3$
B. $e^{3x} + 3$
C. e^{8x+9}
D. $3(x + 1)$
E. $\log_e(3x) + 3$

$$g(f(x)) = \log_e(2f(x)) + 3$$
$$= \log_e(e^{3x}) + 3$$
$$= 3x + 3$$

SECTION 1 – continued

Question 6

A function g with domain R has the following properties.

- $g'(x) = x^2 - 2x$
- the graph of $g(x)$ passes through the point $(1, 0)$

$g(x)$ is equal to

A. $2x - 2$

B. $\frac{x^3}{3} - x^2$

C. $\frac{x^3}{3} - x^2 + \frac{2}{3}$

D. $x^2 - 2x + 2$

E. $3x^3 - x^2 - 1$

$g'(x) = x^2 - 2x$

$g(x) = \int x^2 - 2x \, dx$

$= \frac{x^3}{3} - x^2 + c$

$g(1) = 0$

$\therefore 0 = \frac{1}{3} - 1 + c$

$\therefore c = \frac{2}{3}$

$g(x) = \frac{x^3}{3} - x^2 + \frac{2}{3}$

Question 7

The simultaneous linear equations $(m-1)x + 5y = 7$ and $3x + (m-3)y = 0.7m$ have infinitely many solutions for

A. $m \in R \setminus \{0, -2\}$
B. $m \in R \setminus \{0\}$
C. $m \in R \setminus \{6\}$
D. $m = 6$
E. $m = -2$

$$\begin{bmatrix} m-1 & 5 \\ 3 & m-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 0.7m \end{bmatrix}$$

$\det(A) = 0 \quad \therefore (m-1)(m-3) - 15 = 0$

$m^2 - 4m - 12 = 0$

$(m-6)(m+2) = 0$

$m = 6, -2$

If $m = 6$:
$5x + 5y = 7$
$3x + 3y = 0.42$
Both are the Same: $x + y = 0.14$
so infinitely many solutions

If $m = -2$
$-3x + 5y = 7$
$3x - 5y = -1.4$
Parallel lines
$\downarrow$
$\boxed{\text{NO SOLUTION}}$

Question 8

The function f has rule $f(x) = 3\log_e(2x)$.

If $f(5x) = \log_e(y)$ then y is equal to

A. $30x$
B. $6x$
C. $125x^3$
D. $50x^3$
E. $1000x^3$

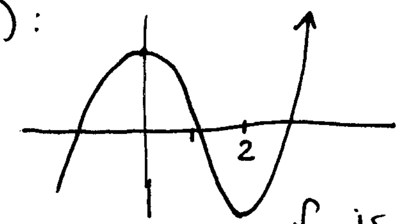
$f(5x) = 3\log_e(10x)$
$= \log_e((10x)^3)$
$= \log_e(1000x^3)$

Question 9

The function $f: (-\infty, a] \to R$ with rule $f(x) = x^3 - 3x^2 + 3$ will have an inverse function provided

A. $a \leq 0$
B. $a \geq 2$
C. $a \geq 0$
D. $a \leq 2$
E. $a \leq 1$

Graph $f(x)$:



f is one to one if $a \leq 0$
(domain must not extend beyond a turning point)

Question 10

The average value of the function $f(x) = e^{2x}\cos(3x)$ for $0 \leq x \leq \pi$ is closest to

A. -82.5
B. 26.3
C. -26.3
D. -274.7
E. π

$$\frac{1}{\pi - 0}\int_0^{\pi} e^{2x}\cos(3x)\,dx$$

≈ -26.3

Question 11

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases}\cos(2x) & \text{if } \frac{3\pi}{4}<x<\frac{5\pi}{4}\\ 0 & \text{elsewhere}\end{cases}$$

The value of a such that $\Pr(X < a) = 0.25$ is closest to

A. 2.25
B. 2.75
C. 2.88
D. 3.06
E. 3.41

$\Pr(X<a) = 0.25 \quad \therefore \int_{\frac{3\pi}{4}}^{a} \cos(2x)\,dx = 0.25$

where $\frac{3\pi}{4} < a < \frac{5\pi}{4}$

Need to type domain

$\therefore a \simeq 2.88$

Question 12

A soccer player is practising her goal kicking. She has a probability of $\frac{3}{5}$ of scoring a goal with each attempt. She has 15 attempts.

The probability that the number of goals she scores is less than 7 is closest to

A. 0.0612
B. 0.0950
C. 0.1181
D. 0.2131
E. 0.7869

X = no of goals $\quad X \overset{d}{=} Bi(n=15, p=\frac{3}{5})$

$\Pr(X<7) = \text{binomcdf}(\frac{3}{5}, 15, 0, 6)$

$= \Pr(X \leq 6) \simeq 0.0950$

Question 13

The continuous random variable X has a normal distribution with mean 20 and standard deviation 6. The continuous random variable Z has the standard normal distribution.

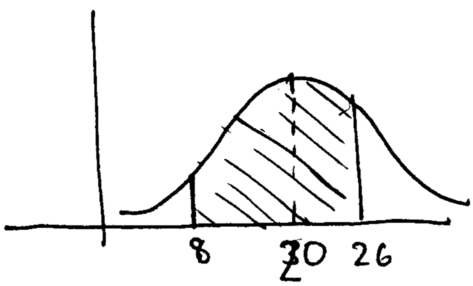
The probability that Z is between -2 and 1 is equal to

A. $\Pr(18 < X < 21)$
B. $\Pr(14 < X < 32)$
C. $\Pr(14 < X < 26)$
D. $\Pr(8 < X < 32)$
E. $\Pr(X > 14) + \Pr(X < 26)$

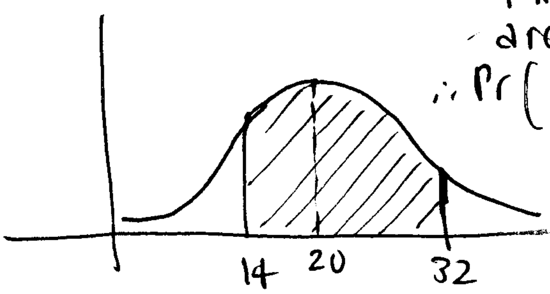
If $z=-2$, $X = 20-12=8$

If $z=1$, $X = 20+6=26$

$\therefore \Pr(-2<z<1) = \Pr(8<X<26)$



These two areas are equal



$\therefore \Pr(8<X<26) = \Pr(14<X<32)$

Question 14

A bag contains four white balls and six black balls. Three balls are drawn from the bag without replacement. The probability that they are all black is

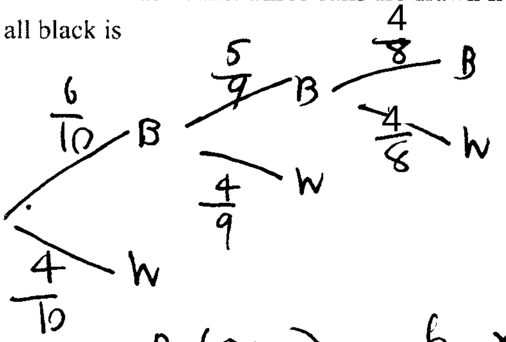
A. $\frac{1}{6}$

B. $\frac{27}{125}$

C. $\frac{24}{29}$

D. $\frac{3}{500}$

E. $\frac{8}{125}$



$$\Pr(BBB) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8}$$
$$= \frac{3}{5} \times \frac{5}{9} \times \frac{1}{2} = \frac{1}{6}$$

Question 15

The discrete random variable X has the following probability distribution.

X	0	1	2
$\Pr(X = x)$	a	b	0.4

If the mean of X is 1 then

A. $a = 0.3$ and $b = 0.1$

B. $a = 0.2$ and $b = 0.2$

C. $a = 0.4$ and $b = 0.2$

D. $a = 0.1$ and $b = 0.5$

E. $a = 0.1$ and $b = 0.3$

$a + b + 0.4 = 1 \quad \therefore a + b = 0.6$

$E(X) = 1$

$\therefore 0 \times a + 1 \times b + 2 \times 0.4 = 1$

$\therefore b + 0.8 = 1$

$\therefore b = 0.2$

and $a = 0.4$

Question 16

The gradient of the function $f: R \to R, f(x) = \frac{5x}{x^2+3}$ is negative for

A. $-\sqrt{3} < x < \sqrt{3}$

B. $x > 3$

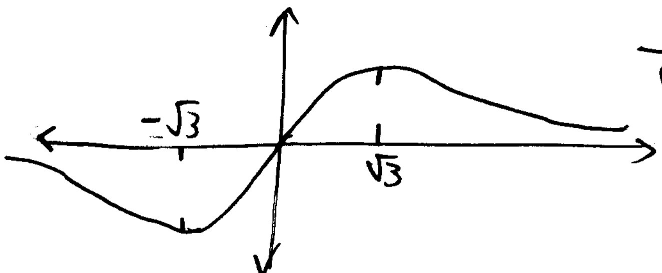
C. $x \in R$

D. $x < -\sqrt{3}$ and $x > \sqrt{3}$

E. $x < 0$

Graph $y = \frac{5x}{x^2+3}$

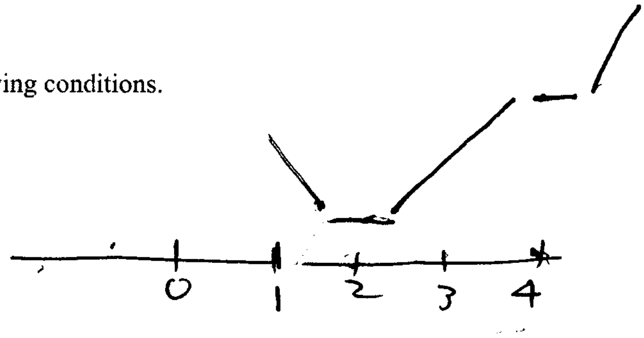
$\frac{dy}{dx} = 0$ if $x = \pm\sqrt{3}$



$\therefore f'(x) < 0$ for $x > \sqrt{3} \cup x < -\sqrt{3}$

Question 17

The function f is differentiable for all $x \in R$ and satisfies the following conditions.



- $f'(x) < 0$ where $x < 2$
- $f'(x) = 0$ where $x = 2$
- $f'(x) = 0$ where $x = 4$
- $f'(x) > 0$ where $2 < x < 4$
- $f'(x) > 0$ where $x > 4$

Which one of the following is true?

A. The graph of f has a local maximum point where $x = 4$.

B. The graph of f has a stationary point of inflection where $x = 4$.

C. The graph of f has a local maximum point where $x = 2$.

D. The graph of f has a local minimum point where $x = 4$.

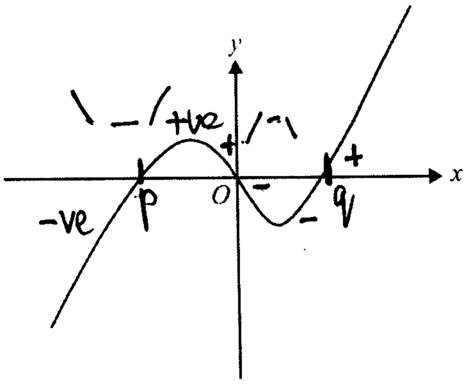
E. The graph of f has a stationary point of inflection where $x = 2$.

Question 18

This question involves modulus so is no longer on the syllabus.

Question 19

The graph of the gradient function $y = f'(x)$ is shown below.



$f'(x)$ is a cubic, so $f(x)$ will be a quartic function

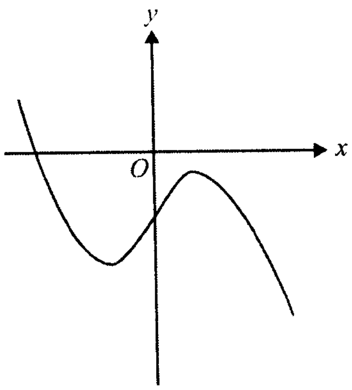
$\therefore$ Eliminate A, B, C

f will have a minimum t/p at $x = p$ and $x = q$ and a max t/p at $x = 0$

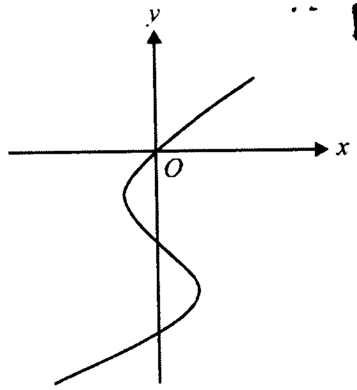
$\therefore$ D is correct

Which of the following could represent the graph of the function $f(x)$?

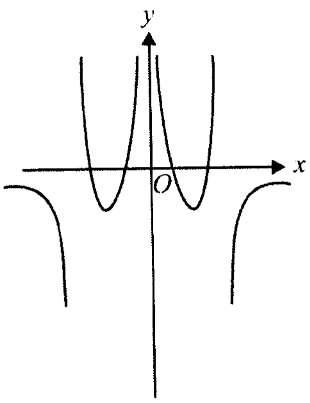
~~**A.**~~



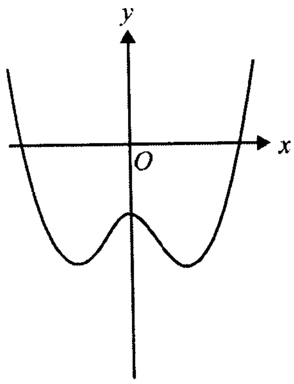
~~**B.**~~



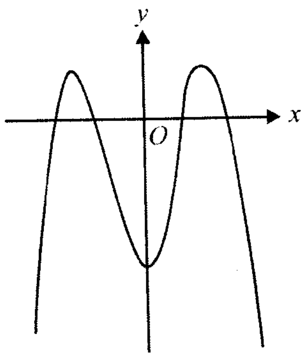
~~**C.**~~



D.



E.



Question 20

Let f be a differentiable function defined for all real x, where $f(x) \geq 0$ for all $x \in [0, a]$.

If $\int_0^a f(x)dx = a$, then $2\int_0^{5a}\left(f\left(\frac{x}{5}\right)+3\right)dx$ is equal to

A. $2a + 6$

B. $10a + 6$

C. $20a$

D. $40a$

E. $50a$

$f\left(\frac{x}{5}\right)$ is dilated by a factor of 5 away from y-axis

$\therefore (a,0) \rightarrow (5a,0)$ and areas will be increased by a factor of 5.

$\therefore 2\int_0^{5a} f\left(\frac{x}{5}\right) + 3\,dx$

$= 2\int_0^{5a} f\left(\frac{x}{5}\right)dx + \int_0^{5a} 6\,dx$

$= 2 \times 5a + \left[6x\right]_0^{5a}$

$= 10a + 30a$

$= 40a$

Question 21

Events A and B are mutually exclusive events of a sample space with

$\Pr(A) = p$ and $\Pr(B) = q$ where $0 < p < 1$ and $0 < q < 1$.

$\Pr(A' \cap B')$ is equal to

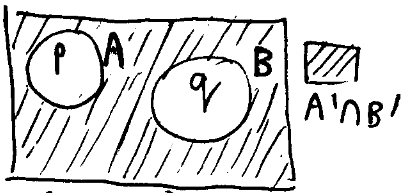
A. $(1-p)(1-q)$

B. $1 - pq$

C. $1 - (p + q)$

D. $2 - p - q$

E. $1 - (p + q - pq)$



$\Pr(A' \cap B') = 1 - (p + q)$

$= 1 - p - q$

Question 22

Let f be a differentiable function defined for $x > 2$ such that

$$\int_3^{ab+2} f(x)dx = \int_3^{a+2} f(x)dx + \int_3^{b+2} f(x)dx \text{ where } a > 1 \text{ and } b > 1.$$

The rule for $f(x)$ is

A. $\sqrt{x-2}$

B. $\log_e(x-2)$

C. $\sqrt{2x-4}$

D. $\log_e|2x-4|$

E. $\frac{1}{x-2}$

We see that addition is somehow being related to multiplication, so this suggests a log function, which is the antiderivative of $\frac{1}{x-2}$. So, we test option E first:

$\int_3^{ab+2} \frac{1}{x-2}\,dx$

$= \left[\log_e(x-2)\right]_3^{ab+2}$

$= \log_e(ab) - \log_e 1$

$= \log_e(ab)$

$= \log_e a + \log_e b$

$\int_3^{a+2} \frac{1}{x-2}\,dx + \int_3^{b+2} \frac{1}{x-2}\,dx$

$= \left[\log_e(x-2)\right]_3^{a+2} + \left[\log_e(x-2)\right]_3^{b+2}$

$= \log_e a + \log_e b$

so this is the right option.

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

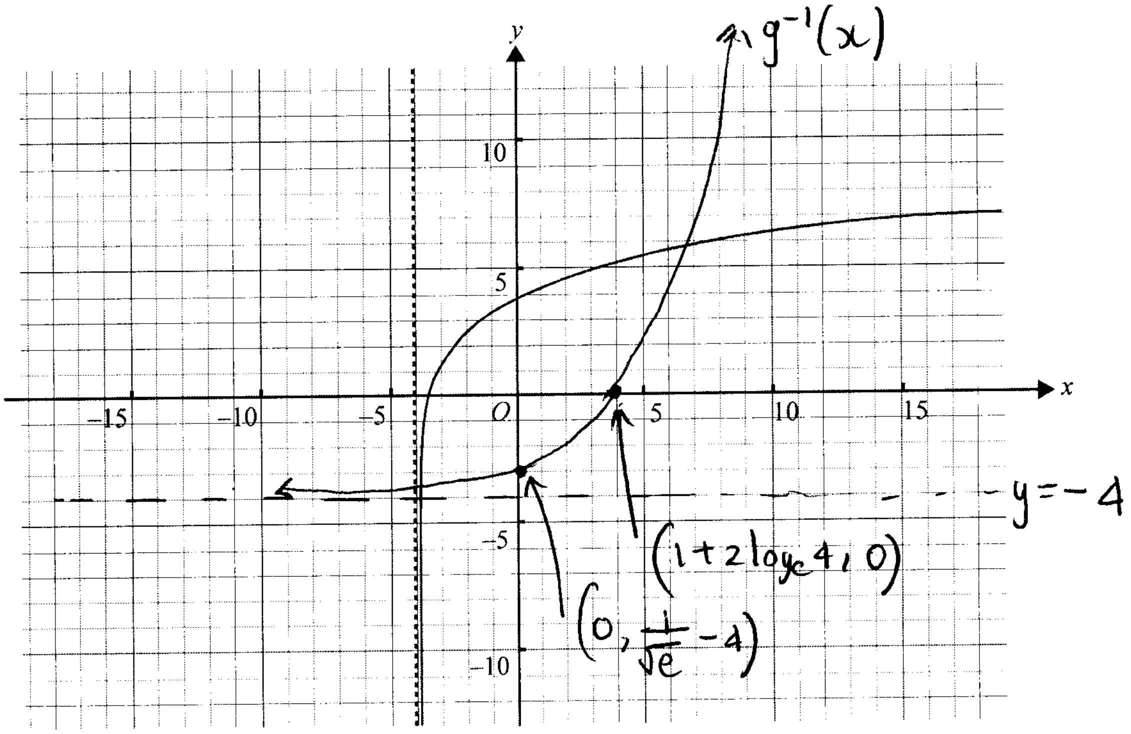
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

a. Part of the graph of the function $g: (-4, \infty) \to R$, $g(x) = 2\log_e(x+4) + 1$ is shown on the axes below.



i. Find the rule and domain of g^{-1}, the inverse function of g.

dom(g)	ran(g)
$(-4, \infty)$	R
dom(g^{-1})	ran(g^{-1})
R	$(-4, \infty)$

$x = 2\log_e(y+4) + 1$

$\frac{x-1}{2} = \log_e(y+4)$

$\therefore g^{-1}(x) = e^{\frac{x-1}{2}} - 4, \quad x \in R$

ii. On the set of axes above sketch the graph of g^{-1}. Label the axes intercepts with their exact values.

x-int: let $y=0$ $\quad e^{\frac{x-1}{2}} - 4 = 0$

$\therefore \frac{x-1}{2} = \log_e 4$

$x = 1 + 2\log_e 4$

y-int: let $x=0$

$y = e^{-\frac{1}{2}} - 4$

iii. Find the values of x, correct to three decimal places, for which $g^{-1}(x) = g(x)$.

Curves intersect on $y = x$

Solving:

$x = 2\log_e(x+4) + 1$ gives

$x \approx -3.914,\ 5.503$

iv. Calculate the area enclosed by the graphs of g and g^{-1}. Give your answer correct to two decimal places.

$$\int_{-3.91432}^{5.50327} \left(2\log_e(x+4)+1\right) - \left(e^{\frac{x-1}{2}} - 4\right)\,dx$$

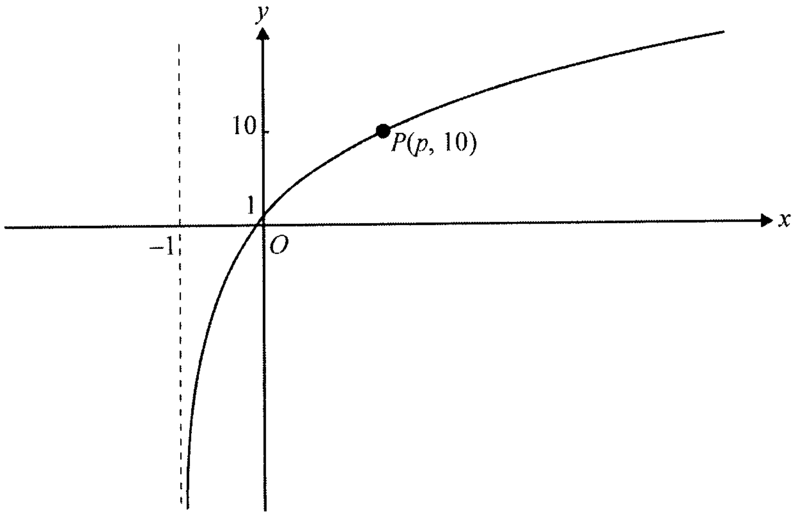
≈ 52.63 sq. units

3 + 3 + 2 + 2 = 10 marks

b. The diagram below shows part of the graph of the function with rule

$$f(x) = k \log_e(x + a) + c, \text{ where } k, a \text{ and } c \text{ are real constants.}$$

- The graph has a vertical asymptote with equation $x = -1$.
- The graph has a y-axis intercept at 1.
- The point P on the graph has coordinates $(p, 10)$, where p is another real constant.



i. State the value of a.

$a = 1$

ii. Find the value of c.

$f(0)=1 \qquad f(x) = k\log_e(x+1) + c$

$\therefore \quad 1 = k\log_e 1 + c$

$\therefore c = 1$

iii. Show that $k = \dfrac{9}{\log_e(p+1)}$.

$f(x) = k\log_e(x+1) + 1$

$f(p) = 10$

$\therefore 10 = k\log_e(p+1) + 1$

$\therefore 9 = k\log_e(p+1)$

$k = \dfrac{9}{\log_e(p+1)}$

SECTION 2 – **Question 1** – continued

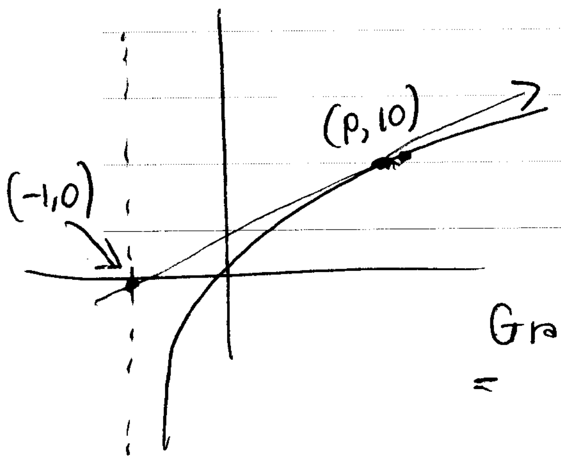
iv. Show that the gradient of the tangent to the graph of f at the point P is $\dfrac{9}{(p+1)\log_e(p+1)}$.

$$f(x) = \frac{9}{\log_e(p+1)} \cdot \log_e(x+1) + 1$$

$$\therefore f'(x) = \frac{9}{\log_e(p+1)} \cdot \frac{1}{x+1}$$

$$\therefore f'(p) = \frac{9}{(p+1)\log_e(p+1)}$$

v. If the point (–1, 0) lies on the tangent referred to in **part b.iv.**, find the exact value of p.



1 + 1 + 2 + 1 + 2 = 7 marks

Total 17 marks

Gradient of tangent

$$= \frac{10-0}{p--1} = \frac{10}{p+1}$$

$$\therefore \frac{10}{p+1} = \frac{9}{(p+1)\log_e(p+1)}$$

$$\therefore 10 = \frac{9}{\log_e(p+1)} \quad (p > -1)$$

$$\therefore \log_e(p+1) = \frac{9}{10}$$

$$p+1 = e^{\frac{9}{10}}$$

$$\therefore p = e^{\frac{9}{10}} - 1$$

Q2 is almost entirely related to Markov Chains, so it is no longer on the syllabus.

Victoria hears that another company, Shoddy Ltd, is producing similar statues (also classified as Superior or Regular), but its statues are entirely made by machines, on a construction line. The quality of any one of Shoddy's statues is independent of the quality of any of the others on its construction line. The probability that any one of Shoddy's statues is Regular is 0.8.

Shoddy Ltd wants to ensure that the probability that it produces at least two Superior statues in a day's production run is at least 0.9.

e. Calculate the minimum number of statues that Shoddy would need to produce in a day to achieve this aim.

Let X = no. of Superior statues $X \overset{d}{=} Bi(n=?, p=0.2)$

$Pr(X \geq 2) \geq 0.9$

$\therefore 1 - Pr(X=0) - Pr(X=1) \geq 0.9$

$\therefore 0.1 \geq Pr(X=0) + Pr(X=1)$

First solve:

$\binom{n}{0}(0.2)^0(0.8)^n + \binom{n}{1}(0.2)0.8^{n-1} = 0.1$

$\therefore (0.8)^n + n \times 0.2(0.8)^{n-1} = 0.1$

Solving: $n = 17.95$

3 marks

$n \geq 18$

Minimum no. of statues = 18

Question 3

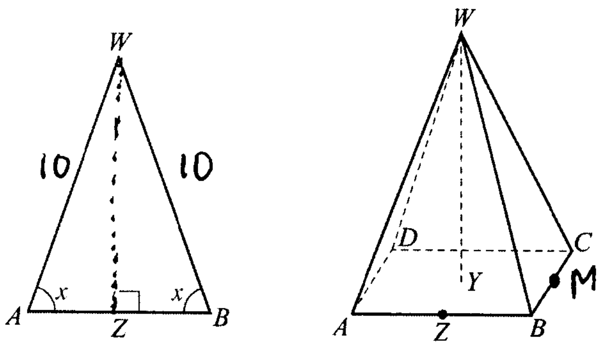
An ancient civilisation buried its kings and queens in tombs in the shape of a square-based pyramid, *WABCD*.

The kings and queens were each buried in a pyramid with $WA = WB = WC = WD = 10$ m.

Each of the isosceles triangle faces is congruent to each of the other triangular faces.

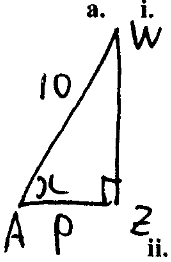
The base angle of each of these triangles is x, where $\frac{\pi}{4} < x < \frac{\pi}{2}$.

Pyramid *WABCD* and a face of the pyramid, *WAB*, are shown here.



Z is the midpoint of AB.

a. **i.** Find AB in terms of x.



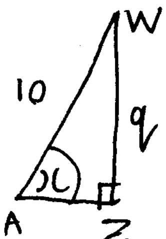
$\cos(x) = \frac{p}{10}$

$p = 10\cos(x)$

$\overline{AB} = 2p$

$\therefore \overline{AB} = 20\cos(x)$

ii. Find WZ in terms of x.



$\sin(x) = \frac{q}{10}$

$q = 10\sin(x)$

$\therefore \overline{WZ} = 10\sin(x)$

1 + 1 = 2 marks

b. Show that the total surface area (including the base), S m^2, of the pyramid, *WABCD*, is given by $S = 400(\cos^2(x) + \cos(x)\sin(x))$.

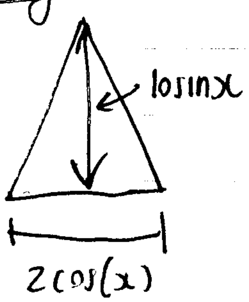
$S = 4\ A_{\text{triangle}} + A_{\text{square}}$

$= 4 \times 10\sin x \cos x + (20\cos x)^2$

$= 400\sin x\cos x + 400\cos^2 x$

$= 400(\cos^2 x + \cos x \sin x)$

Triangle



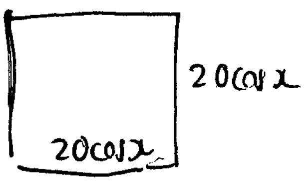
$A = \frac{1}{2} \times 2\cos x \times 10\sin x$

$= 10\sin x \cos x$

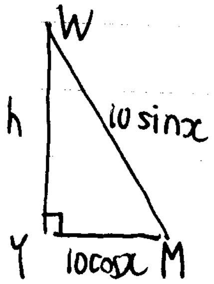
2 marks

SECTION 2 – Question 3 – continued

Square



c. Find WY, the height of the pyramid $WABCD$, in terms of x.



$$h^2 = (10\sin x)^2 - (10\cos x)^2$$

$$h^2 = 100\sin^2 x - 100\cos^2 x$$

$$\therefore h = 10\sqrt{\sin^2 x - \cos^2 x}$$

2 marks

d. The volume of any pyramid is given by the formula Volume $= \frac{1}{3} \times$ area of base $\times$ vertical height.

Show that the volume, T m^3, of the pyramid $WABCD$ is $\frac{4000}{3}\sqrt{\cos^4 x - 2\cos^6 x}$.

$$T = \frac{1}{3} \times 400\cos^2 x \times h$$

$$= \frac{1}{3} \times 400\cos^2 x\sqrt{\sin^2 x - \cos^2 x}$$

$$= \frac{400}{3}\sqrt{\cos^4 x(1 - \cos^2 x - \cos^2 x)}$$ Pythagoriean identity

$$= \frac{400}{3}\sqrt{\cos^4 x - 2\cos^6 x}$$

1 mark

Queen Hepzabah's pyramid was designed so that it had the **maximum possible volume**.

e. Find $\frac{dT}{dx}$ and **hence** find the exact volume of Queen Hepzabah's pyramid and the corresponding value of x.

$$\frac{dT}{dx} = \frac{800\sin x(\cos x)^3}{3\sqrt{1-2\cos^2 x}} - \frac{800\sin x\cos x\sqrt{1-2\cos^2 x}}{3}$$

For a maximum, $\frac{dT}{dx} = 0$ where $\frac{\pi}{4} < x < \frac{\pi}{2}$

Must type domain in when solving.

$$\therefore x = \cos^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

$$T\left(\cos^{-1}\left(\frac{\sqrt{3}}{3}\right)\right) = \frac{400\sqrt{3}}{27} \text{ m}^3$$

4 marks

CAS TIP

Define t(x) and dt(x)=d/dx(t(x)) on CAS.

Queen Hepzabah's daughter, Queen Jepzibah, was also buried in a pyramid. It also had

$$WA = WB = WC = WD = 10 \text{ m}.$$

The volume of Jepzibah's pyramid is exactly one half of the volume of Queen Hepzabah's pyramid. The volume of Queen Jepzibah's pyramid is also given by the formula for T obtained in **part d.**

f. Find the possible values of x, for Jepzibah's pyramid, correct to two decimal places.

Let

$T(x) = \frac{200\sqrt{3}}{27}$

Solving: for $\frac{\pi}{4} < x < \frac{\pi}{2}$

gives:

$x = 0.81, 1.23$

2 marks

Total 13 marks

$x = 0.81, 1.23$

Question 4

Consider the function $f: R \rightarrow R, f(x) = \frac{1}{27}(2x-1)^3(6-3x)+1$.

a. Find the x-coordinate of each of the stationary points of f and state the nature of each of these stationary points.

Let $f'(x) = 0$

$x = \frac{1}{2}, \frac{13}{8}$

	$\frac{1}{2}$		$\frac{13}{8}$	
+	zero	+	zero	−
/	—	/	—	\

At $x = \frac{1}{2}$: stationary point of inflexion

At $x = \frac{13}{8}$: maximum turning point

4 marks

In the following, f is the function $f: R \rightarrow R, f(x) = \frac{1}{27}(ax-1)^3(b-3x)+1$ where a and b are real constants.

b. Write down, in terms of a and b, the possible values of x for which $(x, f(x))$ is a stationary point of f.

Let $f'(x) = 0$

CAS TIP

Define f(x) and df(x) =d/dx(f(x))

$x = \frac{ab+1}{4a}$ or $x = \frac{1}{a}$

3 marks

c. For what value of a does f have no stationary points?

$a = 0$

1 mark

d. Find a in terms of b if f has one stationary point.

If there is only one stationary point,

$$\frac{ab+1}{4a} = \frac{1}{a}$$

$$\therefore \frac{ab+1}{4} = 1$$

$$\therefore ab = 3$$

$$a = \frac{3}{b}$$

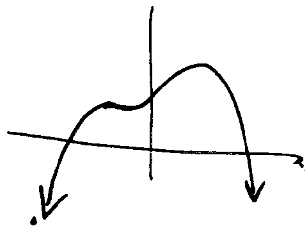
2 marks

e. What is the maximum number of stationary points that f can have?

2

1 mark

(This is a quartic with a cubic component, which has a shape:



f. Assume that there is a stationary point at (1, 1) and another stationary point (p, p) where $p \neq 1$. Find the value of p.

$$f(x) = \frac{1}{27}(ax-1)^3(b-3x)+1$$

From above, we know that 1 stationary point will be at $x = \frac{1}{a}$

$$f\left(\frac{1}{a}\right) = \frac{1}{27} \times 0 + 1 = 1$$

$\therefore \left(\frac{1}{a}, 1\right)$ will always be a stationary point

$\therefore (1, 1) = \left(\frac{1}{a}, 1\right) \quad \therefore a = 1$

$\therefore f(x) = \frac{1}{27}(x-1)^3(b-3x)+1$

Let $f'(x) = 0$

$x = 1$ or $x = \frac{b+1}{4}$

$\therefore f\left(\frac{b+1}{4}\right) = \frac{b+1}{4}$

3 marks

Total 14 marks

Solving: $b = 3$ or 15

If $b = 3$ then we get: $f(1) = 1$

and there is only one stationary point

$\therefore b = 15.$

$$\therefore p = \frac{15+1}{4} = 4$$

END OF QUESTION AND ANSWER BOOK