SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The midpoint of the line segment joining $(0, -5)$ to $(d, 0)$ is

A. $\left(\frac{d}{2}, -\frac{5}{2}\right)$

B. $(0, 0)$

C. $\left(\frac{d-5}{2}, 0\right)$

D. $\left(0, \frac{5-d}{2}\right)$

E. $\left(\frac{5+d}{2}, 0\right)$

Midpoint $= \left(\frac{0+d}{2}, \frac{-5+0}{2}\right)$

$= \left(\frac{d}{2}, \frac{-5}{2}\right)$

Question 2

The gradient of a line **perpendicular** to the line which passes through $(-2, 0)$ and $(0, -4)$ is

A. $\frac{1}{2}$

B. -2

C. $-\frac{1}{2}$

D. 4

E. 2

$m = \frac{-4-0}{0--2} = \frac{-4}{2} = -2$

$\therefore m_{perp} = \frac{1}{2}$

Question 3

If $x + a$ is a factor of $4x^3 - 13x^2 - ax$, where $a \in R\backslash\{0\}$, then the value of a is

A. -4

B. -3

C. -1

D. 1

E. 2

$P(-a) = 0$

$\therefore -4a^3 - 13a^2 + a^2 = 0$

$\therefore -4a^3 - 12a^2 = 0$

$-4a^2(a+3) = 0$

$a = -3$

Question 4

The derivative of $\log_e(2f(x))$ with respect to x is

A. $\dfrac{f'(x)}{f(x)}$

B. $2\dfrac{f'(x)}{f(x)}$

C. $\dfrac{f'(x)}{2f(x)}$

D. $\log_e(2f'(x))$

E. $2\log_e(2f'(x))$

$$\frac{d}{dx}\left[\log_e\left(2(f(x))\right)\right]$$

$$= \frac{1}{2f(x)} \times 2f'(x)$$

$$= \frac{f'(x)}{f(x)}$$

Question 5

The inverse function of $g: [2, \infty) \to R,\ g(x) = \sqrt{2x-4}$ is

A. $g^{-1}: [2, \infty) \to R,\ g^{-1}(x) = \dfrac{x^2+4}{2}$

B. $g^{-1}: [0, \infty) \to R,\ g^{-1}(x) = (2x-4)^2$

C. $g^{-1}: [0, \infty) \to R,\ g^{-1}(x) = \sqrt{\dfrac{x}{2}+4}$

D. $g^{-1}: [0, \infty) \to R,\ g^{-1}(x) = \dfrac{x^2+4}{2}$

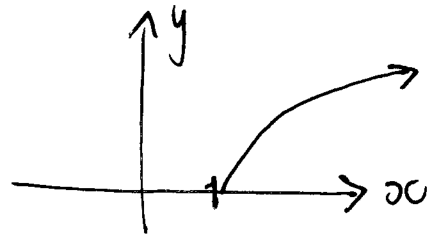
E. $g^{-1}: R \to R,\ g^{-1}(x) = \dfrac{x^2+4}{2}$

$y = \sqrt{2x-4}$

$x = \sqrt{2y-4}$

$x^2 + 4 = 2y$

$y = \dfrac{x^2}{2} + \dfrac{4}{2}$



dom(f)	ran(f)
$[2,\infty)$	$[0,\infty)$
dom(f^{-1})	ran(f^{-1})
$[0,\infty)$	$[2,\infty)$

Question 6

For the continuous random variable X with probability density function

$$f(x) = \begin{cases} \log_e(x) & 1 \le x \le e \\ 0 & elsewhere \end{cases}$$

the expected value of X, $E(X)$, is closest to

A. 0.358

B. 0.5

C. 1

D. 1.859

E. 2.097

$$E(X) = \int_1^e x \log_e x \, dx \approx 2.097$$

Question 7

This question relates to the Linear Approximation and is no longer on the course

Question 8

Consider the function $f: R \to R, f(x) = x(x-4)$ and the function

$$g: \left[\frac{3}{2}, 5\right) \to R, g(x) = x + 3.$$

If the function $h = f + g$, then the domain of the inverse function of h is

A. $[0, 13)$

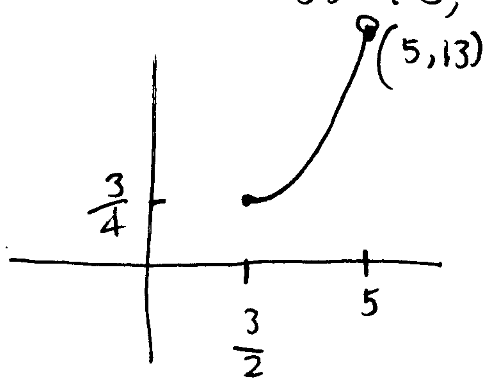
B. $\left[-\frac{3}{4}, 10\right]$

C. $\left(-\frac{3}{4}, \frac{15}{4}\right]$

D. $\left[\frac{3}{4}, 13\right)$

E. $\left[\frac{3}{2}, 13\right)$

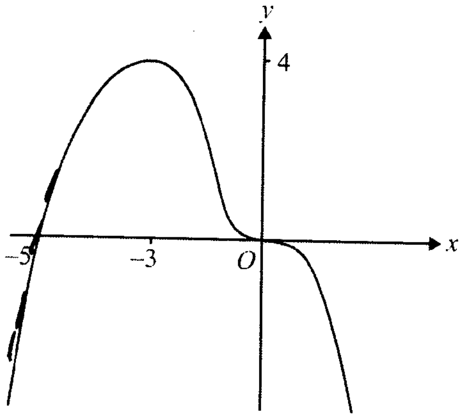
$$f+g = x(x-4) + x + 3, \quad \frac{3}{2} \leq x < 5$$
$$= x^2 - 4x + x + 3$$
$$= x^2 - 3x + 3, \quad \frac{3}{2} \leq x < 5$$



dom(h)	ran(h)
$\left[\frac{3}{2}, 5\right)$	$\left[\frac{3}{4}, 13\right)$
dom(h^{-1})	ran(h^{-1})
$\left[\frac{3}{4}, 13\right)$	$\left[\frac{3}{2}, 5\right)$

Question 9

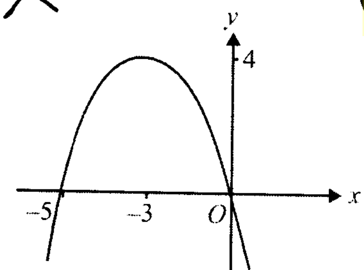
The graph of the function $y = f(x)$ is shown below.



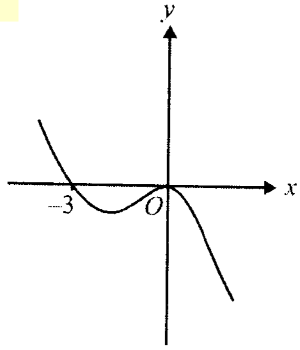
f '(x) must be a cubic with x-intercepts at (−3,0) and (0,0)
B has the correct orientation ($f'(x) > 0$ for $x < -3$)

Which of the following could be the graph of the derivative function $y = f'(x)$?

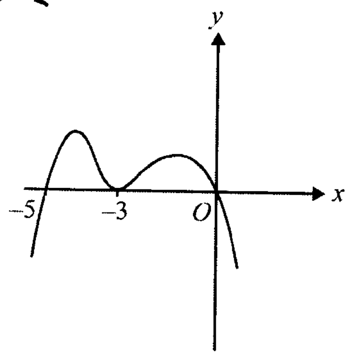
~~A.~~



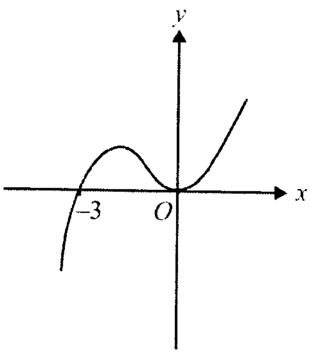
B.



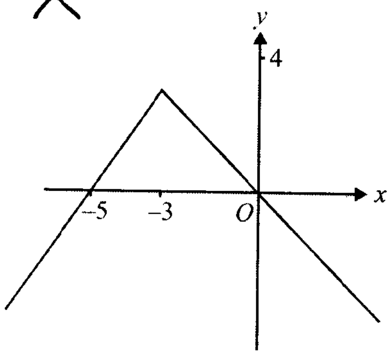
~~C.~~



D.



~~E.~~



Question 10

This question relates to Markov chains and is no longer on the course

Question 11

The average value of the function with rule $f(x) = \log_e(x + 2)$ over the interval [0, 3] is

A. $\log_e(2)$

B. $\frac{1}{3}\log_e(6)$

C. $\log_e\left(\frac{3125}{4}\right) - 3$

D. $\frac{1}{3}\log_e\left(\frac{3125}{4}\right) - 3$

E. $\frac{5\log_e(5) - 2\log_e(2) - 3}{3}$

$$\frac{1}{3-0}\int_0^3 \log_e(x+2)\,dx$$

$$= \frac{1}{3}\int_0^3 \log_e(x+2)\,dx$$

$$= \frac{5\log_e 5 - 2\log_e 2 - 3}{3}$$

Question 12

The continuous random variable X has a normal distribution with mean 30 and standard deviation 5. For a given number a, $\Pr(X > a) = 0.20$.

Correct to two decimal places, a is equal to

A. 23.59

B. 24.00

C. 25.79

D. 34.21

E. 36.41

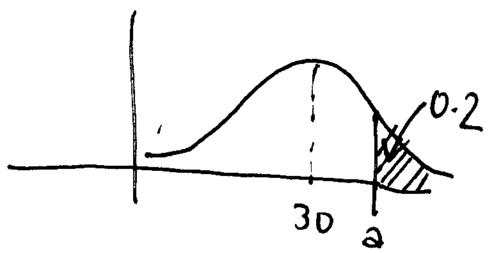
$X \overset{d}{=} N(\mu = 30, \sigma = 5)$

$\Pr(X > a) = 0.2$

$\therefore \Pr(X < a) = 0.8$

$a = \text{invNorm}(0.8, 30, 5)$

≈ 34.21



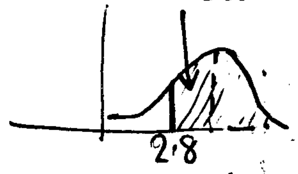
Question 13

In an orchard of 2000 apple trees it is found that 1735 trees have a height greater than 2.8 metres. The heights are distributed normally with a mean μ and standard deviation 0.2 metres.

The value of μ is closest to

A. 3.023
B. 2.577
C. 2.230
D. 1.115
E. 0.223

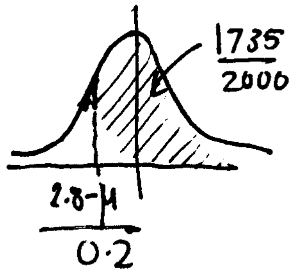
$\frac{1735}{2000}$



$$H \overset{d}{=} N(\mu = ?, \sigma = 0.2)$$

$$\Pr(H > 2.8) = \frac{1735}{2000}$$

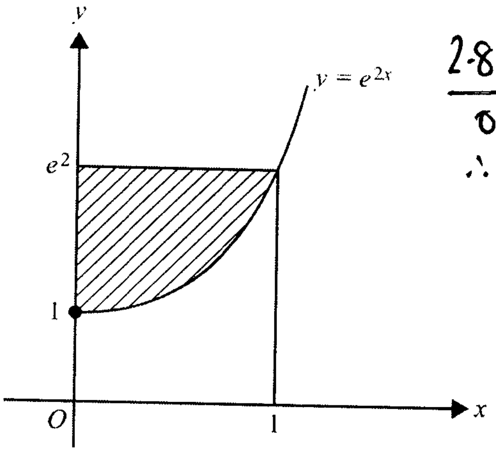
$$\frac{2.8 - \mu}{0.2} = \text{invNorm}(0.1325, 0, 1)$$



$$\frac{2.8 - \mu}{0.2} = -1.11465$$

$$\therefore \mu = 3.023$$

Question 14



To find the area of the shaded region in the diagram shown, four different students proposed the following calculations.

~~**i.**~~ $\int_0^1 e^{2x}\,dx$

ii. $e^2 - \int_0^1 e^{2x}\,dx$ ✓

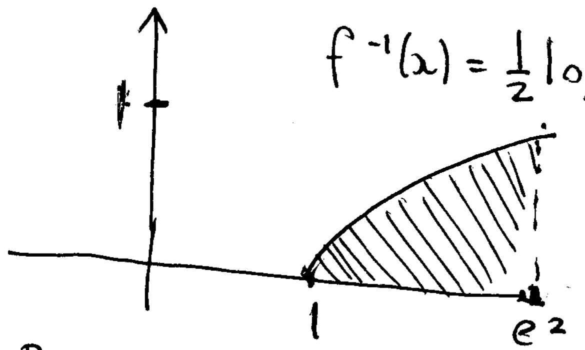
~~**iii.**~~ $\int_1^{e^2} e^{2y}\,dy$

iv. $\int_1^{e^2} \frac{\log_e(x)}{2}\,dx$ ✓

$$A = A_{\text{rectangle}} - A_{\text{curve \& x axis}}$$

$$= e^2 \times 1 - \int_0^1 e^{2x}\,dx$$

$$f^{-1}(x) = \frac{1}{2}\log_e x$$



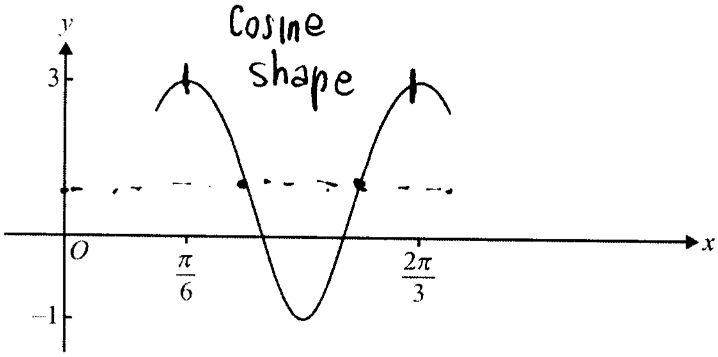
Which of the following is correct?

A. **ii.** only
B. **ii.** and **iii.** only
C. **i.**, **ii.**, **iii.** and **iv.**
D. **ii.** and **iv.** only
E. **i.** and **iv.** only

By symmetry, the area is also equal to:

$$\int_1^{e^2} \frac{1}{2}\log_e x\,dx$$

Question 15



The graph shown could have equation

A. $y = 2\cos\left(x + \frac{\pi}{6}\right) + 1$

B. $y = 2\cos 4\left(x - \frac{\pi}{6}\right) + 1$

C. $y = 4\sin 2\left(x - \frac{\pi}{12}\right) - 1$

D. $y = 3\cos\left(2x + \frac{\pi}{6}\right) - 1$

E. $y = 2\sin\left(4x + \frac{2\pi}{3}\right) - 1$

Period $= \frac{4\pi}{6} - \frac{\pi}{6} = \frac{\pi}{2}$

$\therefore \frac{2\pi}{n} = \frac{\pi}{2} \quad \therefore n = 4$

Vertical shift : $\frac{3 + -1}{2} = 1$

Amplitude : 2

Horizontal shift: $\frac{\pi}{6}$ to Right

$3 \uparrow +2$, 1, $-1 \downarrow -2$

Question 16

This question relates to modulus functions and is no longer on the course

The normal to the curve with equation $y = x^{\frac{3}{2}} + x$ at the point (4, 12) is parallel to the straight line with equation

A. $4x = y$

B. $4y + x = 7$

C. $y = \frac{x}{4} + 1$

D. $x - 4y = -5$

E. $4y + 4x = 20$

$\frac{dy}{dx} = \frac{3}{2}x^{1/2} + 1$

$\left.\frac{dy}{dx}\right|_{x=4} = \frac{3}{2} \times \sqrt{4} + 1$

$= 4$

$\therefore m_{NORMAL} = -\frac{1}{4}$

$\therefore 4y + x = 7$

$y = -\frac{x}{4} + \frac{7}{4}$ would be parallel

Question 18

The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when

A. $-7 < w < 25$

B. $w \leq -7$

C. $w \geq 25$

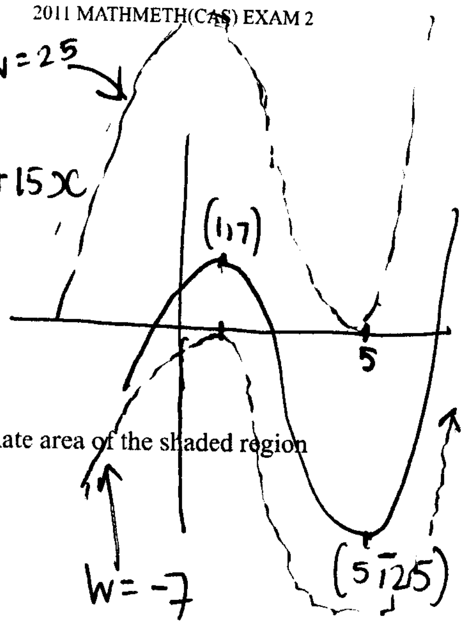
D. $w < -7$ or $w > 25$

E. $w > 1$

Graph $y = x^3 - 9x^2 + 15x$

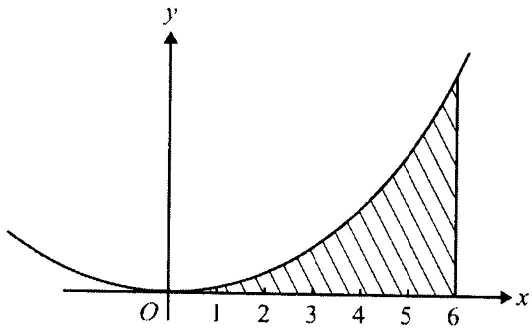
For one x-intercept:

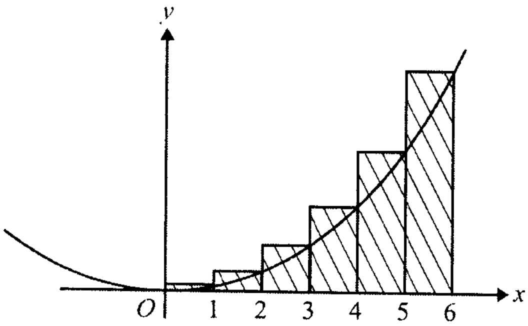
$w > 25$ or $w < -7$



Question 19

A part of the graph of $f: R \to R, f(x) = x^2$ is shown below. Zoe finds the approximate area of the shaded region by drawing rectangles as shown in the second diagram.





Zoe's approximation is p% more than the exact value of the area.

The value of p is closest to

A. 10

B. 15

C. 20

D. 25

E. 30

$$\int_0^6 x^2\,dx = \left[\frac{x^3}{3}\right]_0^6 = \frac{6^3}{3} = 72$$

Approximation:

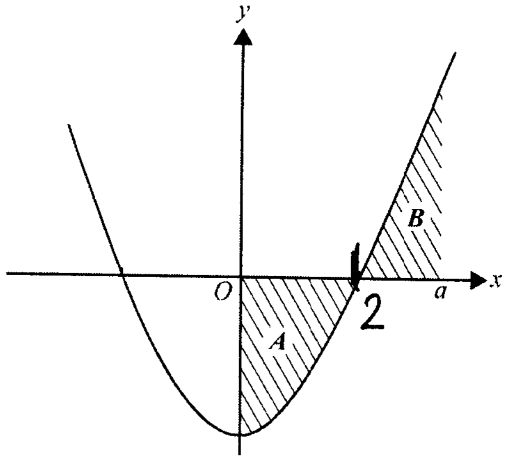
$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$
$$= 1 + 4 + 9 + 16 + 25 + 36$$
$$= 91$$

$$91 - 72 = 19$$

$$p \simeq \frac{19}{72} \times 100\% \simeq 26\%$$

Question 20

A part of the graph of $g: R \to R$, $g(x) = x^2 - 4$ is shown below.



The area of the region marked ***A*** is the same as the area of the region marked ***B***.

The exact value of a is

A. 0
B. 6
C. $\sqrt{6}$
D. 12
E. $2\sqrt{3}$

$$\int_0^a x^2 - 4\,dx = 0 \quad \therefore \left[\frac{x^3}{3} - 4x\right]_0^a = 0$$

$$\frac{a^3}{3} - 4a = 0$$

$$\therefore \frac{a^2}{3} - 4 = 0$$

$$\therefore a = \sqrt{12} \ (a > 0)$$

$$\therefore a = 2\sqrt{3}$$

Question 21

For two events, P and Q, $\Pr(P \cap Q) = \Pr(P' \cap Q)$.

P and Q will be independent events exactly when

A. $\Pr(P') = \Pr(Q)$
B. $\Pr(P \cap Q') = \Pr(P' \cap Q)$
C. $\Pr(P \cap Q) = \Pr(P) + \Pr(Q)$
D. $\Pr(P \cap Q') = \Pr(P \cap Q)$
E. $\Pr(P) = \frac{1}{2}$

	Q	Q'	
P	x		
P'	x		
	$2x$	$1-2x$	1

If P and Q are independent

$$\Pr(P \cap Q) = \Pr(P) \times \Pr(Q)$$

$$\therefore x = \Pr(P) \times 2x$$

$$\therefore 1 = 2\Pr(P)$$

$$\therefore \Pr(P) = \frac{1}{2}$$

Question 22

The expression

$$\log_c(a) + \log_a(b) + \log_b(c)$$

is equal to

A. $\dfrac{1}{\log_c(a)} + \dfrac{1}{\log_a(b)} + \dfrac{1}{\log_b(c)}$

B. $\dfrac{1}{\log_a(c)} + \dfrac{1}{\log_b(a)} + \dfrac{1}{\log_c(b)}$

C. $-\dfrac{1}{\log_a(b)} - \dfrac{1}{\log_b(c)} - \dfrac{1}{\log_c(a)}$

D. $\dfrac{1}{\log_a(a)} + \dfrac{1}{\log_b(b)} + \dfrac{1}{\log_c(c)}$

E. $\dfrac{1}{\log_c(ab)} + \dfrac{1}{\log_b(ac)} + \dfrac{1}{\log_a(cb)}$

$$\log_c a = \frac{\log_a a}{\log_a c} = \frac{1}{\log_a c}$$

$$\log_a b = \frac{1}{\log_b a}$$

$$\log_b c = \frac{1}{\log_c b}$$

$$\therefore \log_c a + \log_a b + \log_b c = \frac{1}{\log_a c} + \frac{1}{\log_b a} + \frac{1}{\log_c b}$$

END OF SECTION 1
TURN OVER

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Two ships, the Elsa and the Violet, have collided. Fuel immediately starts leaking from the Elsa into the sea. The captain of the Elsa estimates that at the time of the collision his ship has 6075 litres of fuel on board and he also forecasts that it will leak into the sea at a rate of $\frac{t^2}{5}$ litres per minute, where t is the number of minutes that have elapsed since the collision.

a. At this rate how long, in minutes, will it take for all the fuel from the Elsa to leak into the sea?

$$\frac{dV}{dt} = \frac{-t^2}{5}$$

V = Volume of oil inside the ship

NOTE: It is important to define what the variable V means.

$$V = \int \frac{-t^2}{5}\,dt$$

$$V = \frac{-t^3}{15} + c$$

If V referred to the volume of fuel in the sea, then dv/dt would be positive.

When $t = 0$, $V = 6075$

$\therefore 6075 = 0 + C$

$\therefore C = 6075$

3 marks

$$V = 6075 - \frac{t^3}{15}$$

When $V = 0$, ship is empty

$\therefore 6075 = \frac{t^3}{15}$

$t^3 = 15 \times 6075$

$\therefore t = 45$

$\therefore$ 45 minutes

The rest of Q1 involves related rates and modulus functions and so is no longer on the course.

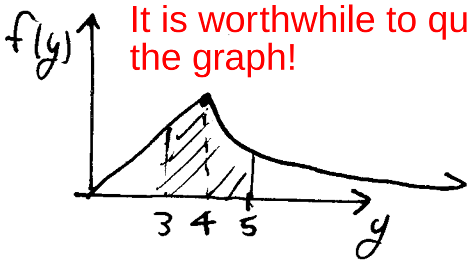
Question 2

In a chocolate factory the material for making each chocolate is sent to one of two machines, machine A or machine B.

The time, X seconds, taken to produce a chocolate by machine A, is normally distributed with mean 3 and standard deviation 0.8.

The time, Y seconds, taken to produce a chocolate by machine B, has the following probability density function.

$$f(y) = \begin{cases} 0 & y < 0 \\ \dfrac{y}{16} & 0 \le y \le 4 \\ 0.25e^{-0.5(y-4)} & y > 4 \end{cases}$$



It is worthwhile to quickly sketch the graph!

a. Find correct to four decimal places

i. $\Pr(3 \le X \le 5)$

$X \stackrel{d}{=} N(\mu = 3, \sigma = 0.8)$

$\Pr(3 \le X \le 5) = 0.4938$

ii. $\Pr(3 \le Y \le 5)$

$\Pr(3 \le Y \le 5) = \int_3^4 \frac{y}{16}\,dy + \int_4^5 0.25e^{-0.5(y-4)}\,dy$

$\simeq 0.4155$

1 + 3 = 4 marks

b. Find the mean of Y, correct to three decimal places.

$E(Y) = \int_0^4 \frac{y^2}{16}\,dy + \int_4^\infty 0.25y\,e^{-0.5(y-4)}\,dy$

$= 4.333$ seconds

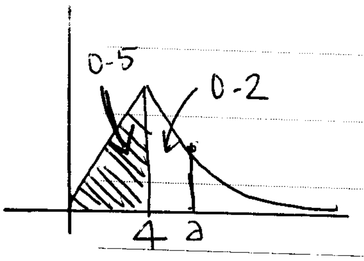
3 marks

c. i. Find the median of Y.

$$\int_0^4 \frac{y}{16}\,dy = \left[\frac{y^2}{32}\right]_0^4 = \frac{16}{32} = \frac{1}{2}$$

$\therefore$ Median $= 4$ seconds

ii. Find the value of a, correct to two decimal places, such that $\Pr(Y \le a) = 0.7$.



$$\int_4^a 0.25\, e^{-0.5(y-4)}\,dy = 0.2$$

Solving for a: $a \approx 5.02$

Again the graph comes in handy!

$1 + 2 = 3$ marks

d. It can be shown that $\Pr(Y \le 3) = \frac{9}{32}$. A random sample of 10 chocolates **produced by machine *B*** is chosen. Find the probability, correct to four decimal places, that exactly 4 of these 10 chocolates took 3 or less seconds to produce.

Let V = no. of chocolates that took 3 or less minutes to produce $V \overset{d}{=} Bi\left(n = 10, p = \frac{9}{32}\right)$

$\Pr(V = 4) = 0.1812$

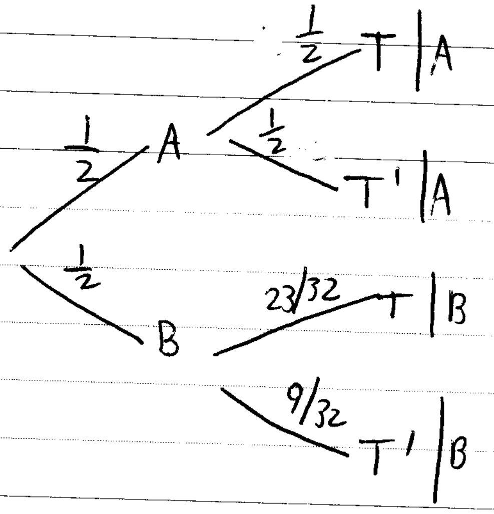
Note the working is the definition of the variable V, the probability statementPr(V=4) and the specification of the distribution type (binomial) and the parameters n, p.

2 marks

All of the chocolates produced by machine A and machine B are stored in a large bin. There is an equal number of chocolates from each machine in the bin.

It is found that if a chocolate, produced by either machine, takes longer than 3 seconds to produce then it can easily be identified by its darker colour.

e. A chocolate is selected at random from the bin. It is found to have taken longer than 3 seconds to produce. Find, correct to four decimal places, the probability that it was produced by machine A.



T = Took longer than 3 min

$Pr(T|A) = Pr($chocolate from A took more than 3 min$)$

$= Pr(X > 3)$ where $X \stackrel{d}{=} N(\mu = 3, \sigma = 0.8)$

$= 0.5$

$$Pr(A|T) = \frac{Pr(A \cap T)}{Pr(T)}$$

$$Pr(T) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{23}{32} = \frac{1}{4} + \frac{23}{64} = \frac{39}{64}$$

$$Pr(A \cap T) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

3 marks

Total 15 marks

$$\therefore Pr(A|T) = \frac{\frac{1}{4}}{\frac{39}{64}}$$

$$= \frac{1}{4} \times \frac{64}{39}$$

$$= \frac{16}{39}$$

$$\simeq 0.4103$$

Tree Diagram was the best strategy for solving this problem

Question 3

a. Consider the function $f: R \to R, f(x) = 4x^3 + 5x - 9$.

i. Find $f'(x)$

$f'(x) = 12x^2 + 5$

ii. Explain why $f'(x) \geq 5$ for all x.

$12x^2 \geq 0$ for all $x \in \mathbb{R}$

$\therefore f'(x) \geq 5$ for all $x \in \mathbb{R}$

1 + 1 = 2 marks

b. The cubic function p is defined by $p: R \to R, p(x) = ax^3 + bx^2 + cx + k$, where a, b, c and k are real numbers.

i. If p has m stationary points, what possible values can m have?

$m = 0, 1, 2$

Any cubic can have 0, 1 or 2 stationary points)

ii. If p has an inverse function, what possible values can m have?

$m = 0, 1$

(if m = 2, then the function will not be one to one)

1 + 1 = 2 marks

c. The cubic function q is defined by $q: R \to R, q(x) = 3 - 2x^3$.

i. Write down an expression for $q^{-1}(x)$.

$y = 3 - 2x^3$

$\downarrow$

$x = 3 - 2y^3$

$x - 3 = -2y^3$

$2y^3 = 3 - x$

$y^3 = \frac{3-x}{2}$

$\therefore q^{-1}(x) = \sqrt[3]{\frac{3-x}{2}}$

ii. Determine the coordinates of the point(s) of intersection of the graphs of $y = q(x)$ and $y = q^{-1}(x)$.

$q(x)$ and $q^{-1}(x)$ intersect on line $y = x$

$\therefore$ Solving $q(x) = x$

gives: $x = 1$

$\therefore (1,1)$ is point of intersection

2 + 2 = 4 marks

d. The cubic function g is defined by $g: R \to R, g(x) = x^3 + 2x^2 + cx + k$, where c and k are real numbers.

i. If g has exactly one stationary point, find the value of c.

$g'(x) = 3x^2 + 4x + c$

For stationary points, $g'(x) = 0$

$\therefore 3x^2 + 4x + c = 0$

For one solution, $\Delta = 0$

$\therefore (4)^2 - 4 \times 3 \times c = 0$

$16 - 12c = 0 \quad \therefore c = \frac{4}{3}$

ii. If this stationary point occurs at a point of intersection of $y = g(x)$ and $y = g^{-1}(x)$, find the value of k.

$g(x) = x^3 + 2x^2 + \frac{4x}{3} + k$

$g'(x) = 3x^2 + 4x + \frac{4}{3} = 3\left(x^2 + \frac{4x}{3} + \frac{4}{9}\right)$

$= 3\left(x + \frac{2}{3}\right)^2$

$g'(x) = 0$ gives: $3\left(x + \frac{2}{3}\right)^2 = 0 \therefore x = \frac{-2}{3}$

Since the stationary point must lie on $y = x$,

the intersection point is $\left(\frac{-2}{3}, \frac{-2}{3}\right)$

$\therefore g\left(\frac{-2}{3}\right) = \frac{-2}{3}$

where $g(x) = x^3 + 2x^2 + \frac{4x}{3} + k$

gives: $k = \frac{-10}{27}$

$\therefore k = \frac{-10}{27}$

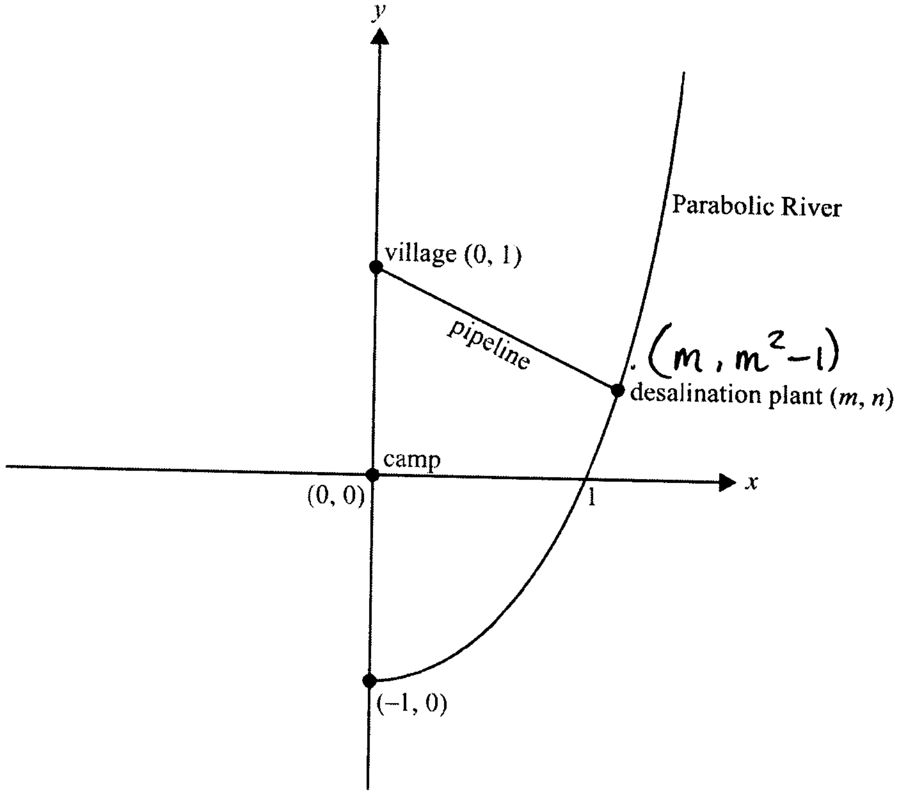
3 + 3 = 6 marks

Total 14 marks

Question 4

Deep in the South American jungle, Tasmania Jones has been working to help the Quetzacotl tribe to get drinking water from the very salty water of the Parabolic River. The river follows the curve with equation $y = x^2 - 1, x \geq 0$ as shown below. All lengths are measured in kilometres.

Tasmania has his camp site at (0, 0) and the Quetzacotl tribe's village is at (0, 1). Tasmania builds a desalination plant, which is connected to the village by a straight pipeline.



a. If the desalination plant is at the point (m, n) show that the length, L kilometres, of the straight pipeline that carries the water from the desalination plant to the village is given by

$$L = \sqrt{m^4 - 3m^2 + 4}.$$

$$L = \sqrt{(m-0)^2 + (m^2-1-1)^2}$$

$$= \sqrt{m^2 + (m^2-2)^2}$$

$$= \sqrt{m^2 + m^4 - 4m^2 + 4}$$

$$= \sqrt{m^4 - 3m^2 + 4}$$

3 marks

b. If the desalination plant is built at the point on the river that is closest to the village

i. find $\frac{dL}{dm}$ and hence find the coordinates of the desalination plant

$$\frac{dL}{dm} = \frac{4m^3 - 6m}{2\sqrt{m^4 - 3m^2 + 4}}$$

For a minimum, $\frac{dL}{dm} = 0$

$\therefore 4m^3 - 6m = 0$

$2m(2m^2 - 3) = 0$

$\therefore m = \sqrt{\frac{3}{2}}$ (since $m > 0$)

$\therefore m = \frac{\sqrt{6}}{2}$ $\therefore$ Co-ordinates: $\left(\frac{\sqrt{6}}{2}, \frac{1}{2}\right)$

ii. find the length, in kilometres, of the pipeline from the desalination plant to the village.

When $m = \frac{\sqrt{6}}{2}$, $l = \frac{\sqrt{7}}{2}$ km

CAS TIP

3 + 2 = 5 marks

(Define $l(m) = \sqrt{m^4 - 3m^2 + 4}$

$l\left(\frac{\sqrt{6}}{2}\right) = \frac{\sqrt{7}}{2}$)

The desalination plant is actually built at $\left(\frac{\sqrt{7}}{2}, \frac{3}{4}\right)$.

If the desalination plant stops working, Tasmania needs to get to the plant in the minimum time.

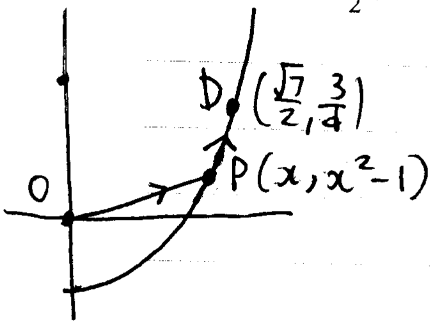
Tasmania runs in a straight line from his camp to a point (x, y) on the river bank where $x \le \frac{\sqrt{7}}{2}$. He then swims up the river to the desalination plant.

Tasmania runs from his camp to the river at 2 km per hour. The time that he takes to swim to the desalination plant is proportional to the difference between the y-coordinates of the desalination plant and the point where he enters the river.

c. Show that the total time taken to get to the desalination plant is given by

This is a "show that" question so reasoning must be shown.

$$T = \frac{1}{2}\sqrt{x^4 - x^2 + 1} + \frac{1}{4}k\left(7 - 4x^2\right)$$ hours where k is a positive constant of proportionality.



Runs from O to P

$$d(OP) = \sqrt{(x-0)^2 + (x^2-1)^2} = \sqrt{x^4 - x^2 + 1}$$

$$\therefore \text{Time taken to run} = \frac{\sqrt{x^4 - x^2 + 1}}{2}$$

Time to swim from P to D

$$= k\left(\frac{3}{4} - (x^2-1)\right)$$

$$= k\left(\frac{7}{4} - x^2\right) = \frac{1}{4}k\left(7 - 4x^2\right)$$

$\therefore$ Total time

$$T(x) = \frac{\sqrt{x^4 - x^2 + 1}}{2} + \frac{k}{4}\left(7 - 4x^2\right)$$

3 marks

The value of k varies from day to day depending on the weather conditions.

d. If $k = \frac{1}{2\sqrt{13}}$

i. find $\frac{dT}{dx}$

$$T(x) = \frac{\sqrt{x^4 - x^2 + 1}}{2} + \frac{1}{8\sqrt{13}}\left(7 - 4x^2\right)$$

$$\therefore \frac{dT}{dx} = \frac{x(2x^2-1)}{2\sqrt{x^4 - x^2 + 1}} - \frac{\sqrt{13}\,x}{13}$$

ii. hence find the coordinates of the point where Tasmania should reach the river if he is to get to the desalination plant in the minimum time.

For a minimum, $\frac{dT}{dx} = 0$

Solving: $x = \frac{\sqrt{3}}{2}$ (since $x > 0$)

When $x = \frac{\sqrt{3}}{2}$, the point $P(x, x^2 - 1)$ is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{4}\right)$

$\therefore$ Required point: $\left(\frac{\sqrt{3}}{2}, -\frac{1}{4}\right)$

1 + 2 = 3 marks

e. On one particular day, the value of k is such that Tasmania should run directly from his camp to the point (1, 0) on the river to get to the desalination plant in the minimum time. Find the value of k on that particular day.

$T(x) = \frac{\sqrt{x^4 - x^2 + 1}}{2} + \frac{k}{4}(7 - 4x^2)$

CAS TIP
Define t(x) on CAS
Define dt(x) = d/dx(t(x))
dt(1)=1/2 - k/2

$T'(1) = 0$

$T'(1) = \frac{1}{2} - 2k$

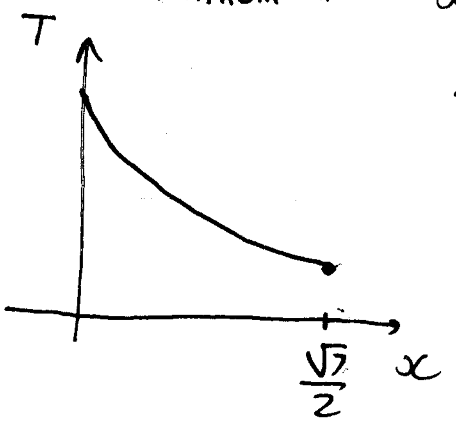
$\therefore \frac{1}{2} - 2k = 0$

$\therefore k = \frac{1}{4}$

2 marks

f. Find the values of k for which Tasmania should run directly from his camp towards the desalination plant to reach it in the minimum time.

This requires that the function $T(x)$ has an endpoint minimum at $x = \frac{\sqrt{7}}{2}$



$\therefore T'\left(\frac{\sqrt{7}}{2}\right) \leq 0$

$\therefore \frac{5\sqrt{259}}{74} - k\sqrt{7} \leq 0$

$\therefore \frac{5\sqrt{259}}{74} \leq \sqrt{7}\,k$

$\therefore k \geq \frac{5\sqrt{259}}{74\sqrt{7}}$

$\therefore k \geq \frac{5\sqrt{37}}{74}$

2 marks

Total 18 marks

END OF QUESTION AND ANSWER BOOK