SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The function with rule $f(x) = -3\sin\left(\frac{\pi x}{5}\right)$ has period

A. 3

B. 5

C. 10

D. $\frac{\pi}{5}$

E. $\frac{\pi}{10}$

$$\text{Period} = \frac{2\pi}{\frac{\pi}{5}} = 10$$

Question 2

For the function with rule $f(x) = x^3 - 4x$, the average rate of change of $f(x)$ with respect to x on the interval $[1, 3]$ is

A. 1

B. 3

C. 5

D. 6

E. 9

$$\frac{f(3) - f(1)}{3-1} = \frac{15 - -3}{2}$$
$$= 9$$

Question 3

The range of the function $f: [-2, 3) \to R,\ f(x) = x^2 - 2x - 8$ is

A. R

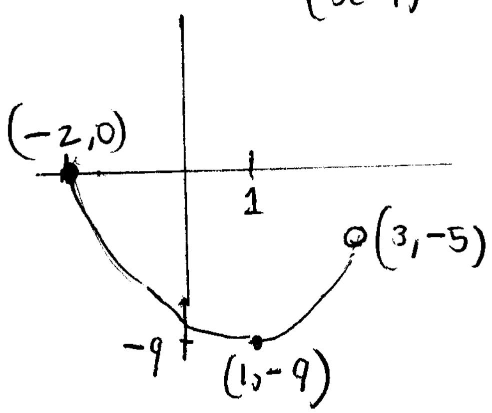
B. $(-9, -5]$

C. $(-5, 0)$

D. $[-9, 0]$

E. $[-9, -5)$

$$f(x) = x^2 - 2x + 1 - 9$$
$$= (x-1)^2 - 9 \quad \therefore T/p: (1, -9)$$



SECTION 1 – continued

Range: $[-9, 0]$

Question 4

Given that g is a differentiable function and k is a real number, the derivative of the composite function $g(e^{kx})$ is

A. $kg'(e^{kx})\,e^{kx}$

B. $kg(e^{kx})$

C. $ke^{kx}\,g(e^{kx})$

D. $ke^{kx}\,g'(e^{x})$

E. $\frac{1}{k}e^{kx}g'(e^{kx})$

Chain Rule:

$$\frac{d}{dx}\{g(e^{kx})\}$$
$$= g'(e^{kx}) \times ke^{kx}$$

Question 5

Let the rule for a function g be $g(x) = \log_e((x-2)^2)$. For the function g, the

A. maximal domain $= R^+$ and range $= R$

B. maximal domain $= R\backslash\{2\}$ and range $= R$

C. maximal domain $= R\backslash\{2\}$ and range $= (-2, \infty)$

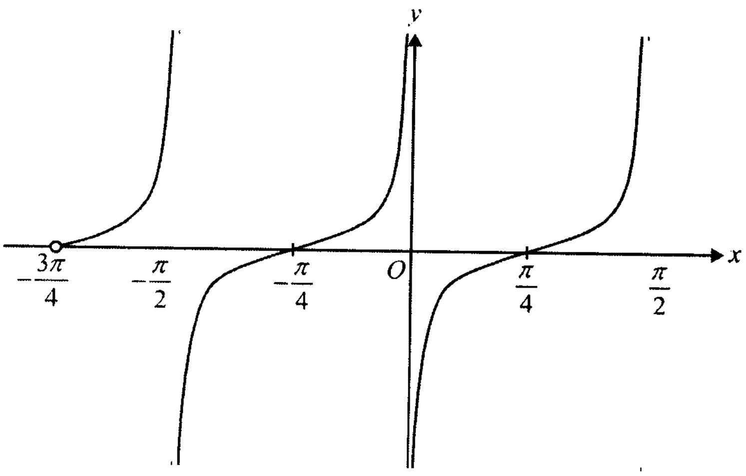
D. maximal domain $= [2, \infty)$ and range $= (0, \infty)$

E. maximal domain $= [2, \infty)$ and range $= [0, \infty)$

$(x-2)^2 > 0$ for $x \in R\backslash\{2\}$

Question 6

A section of the graph of f is shown below.



The rule of f could be

A. $f(x) = \tan(x)$

B. $f(x) = \tan\left(x - \frac{\pi}{4}\right)$

C. $f(x) = \tan\left(2\left(x - \frac{\pi}{4}\right)\right)$

D. $f(x) = \tan\left(2\left(x - \frac{\pi}{2}\right)\right)$

E. $f(x) = \tan\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right)$

Period $= \frac{\pi}{2}$, Translated $\frac{\pi}{4}$ to right

$\therefore y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right)$

Question 7

The temperature, T °C, inside a building t hours after midnight is given by the function

$$f: [0, 24] \to R,\ T(t) = 22 - 10\cos\left(\frac{\pi}{12}(t-2)\right)$$

$t = 2$ $t = 14$

The average temperature inside the building between 2 am and 2 pm is

A. 10 °C
B. 12 °C
C. 20 °C
D. 22 °C
E. 32 °C

Use averge value of a function definition

$$\frac{1}{14-2}\int_2^{14} f(t)\,dt = \frac{1}{12}\int_2^{14} f(t)\,dt = 22$$

Question 8

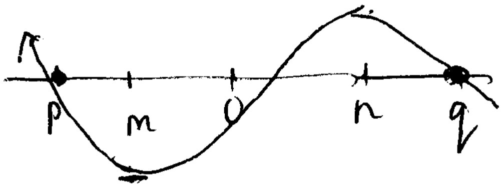
The function $f: R \to R, f(x) = ax^3 + bx^2 + cx$, where a is a negative real number and b and c are real numbers.

For the real numbers $p < m < 0 < n < q$, we have $f(p) = f(q) = 0$ and $f'(m) = f'(n) = 0$.

The gradient of the graph of $y = f(x)$ is negative for

A. $(-\infty, m) \cup (n, \infty)$
B. (m, n)
C. $(p, 0) \cup (q, \infty)$
D. $(p, m) \cup (0, q)$
E. (p, q)

Since $a < 0$ It is a negative cubic



Question 9

The normal to the graph of $y = \sqrt{b - x^2}$ has a gradient of 3 when $x = 1$.

The value of b is

A. $-\frac{10}{9}$

B. $\frac{10}{9}$

C. 4

D. 10

E. 11

$m_{NORMAL} = 3 \quad \therefore m_{TANGENT} = \frac{-1}{3} = -\frac{1}{3}$

$$\frac{dy}{dx} = \frac{1}{2}(b - x^2)^{-1/2} \times -2x = \frac{-x}{\sqrt{b-x^2}}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{-1}{\sqrt{b-1}}$$

$$\therefore \frac{-1}{\sqrt{b-1}} = \frac{-1}{3} \quad \therefore 3 = \sqrt{b-1} \quad \therefore b = 10$$

Question 10

The average value of the function $f: [0, 2\pi] \to R, f(x) = \sin^2(x)$ over the interval $[0, a]$ is 0.4.

The value of a, to three decimal places, is

A. 0.850
B. 1.164
C. 1.298
D. 1.339
E. 4.046

$$\frac{1}{a-0}\int_0^a \sin^2 x\,dx = 0.4$$

$$\therefore \frac{1}{a}\int_0^a \sin^2 x\,dx = 0.4$$

Solving: $a \simeq 1.2979$

Question 11

The weights of bags of flour are normally distributed with mean 252 g and standard deviation 12 g. The manufacturer says that 40% of bags weigh more than x g.

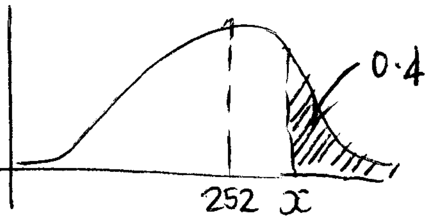
The maximum possible value of x is closest to

A. 249.0
B. 251.5
C. 253.5
D. 254.5
E. 255.0

$\Pr(W > x) = 0.4$

$x = \text{invNorm}(0.6, 252, 12)$

$\therefore x \simeq 255.04$

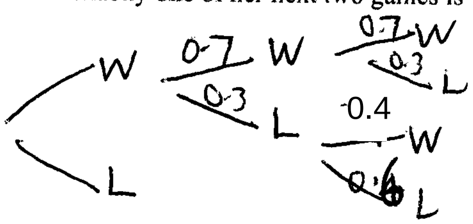


Question 12

Demelza is a badminton player. If she wins a game, the probability that she will win the next game is 0.7. If she loses a game, the probability that she will lose the next game is 0.6. Demelza has just won a game.

The probability that she will win exactly one of her next two games is

A. 0.33
B. 0.35
C. 0.42
D. 0.49
E. 0.82



$\Pr(WL) + \Pr(LW)$

$= 0.7 \times 0.3 + 0.3 \times 0.4$

$= 0.21 + 0.12$

$= 0.33$

Question 13

A and B are events of a sample space S.

$$\Pr(A \cap B) = \frac{2}{5} \text{ and } \Pr(A \cap B') = \frac{3}{7}.$$

$\Pr(B'|A)$ is equal to

A. $\frac{6}{35}$

B. $\frac{15}{29}$

C. $\frac{14}{35}$

D. $\frac{29}{35}$

E. $\frac{2}{3}$

	B	B'	
A	$\frac{2}{5}$	$\frac{3}{7}$	$19/35$
A'			

$\frac{2}{5} + \frac{3}{7}$

$= \frac{14}{35} + \frac{15}{35}$

$= \frac{29}{35}$

$\Pr(B'|A) = \dfrac{\Pr(B' \cap A)}{\Pr(A)}$

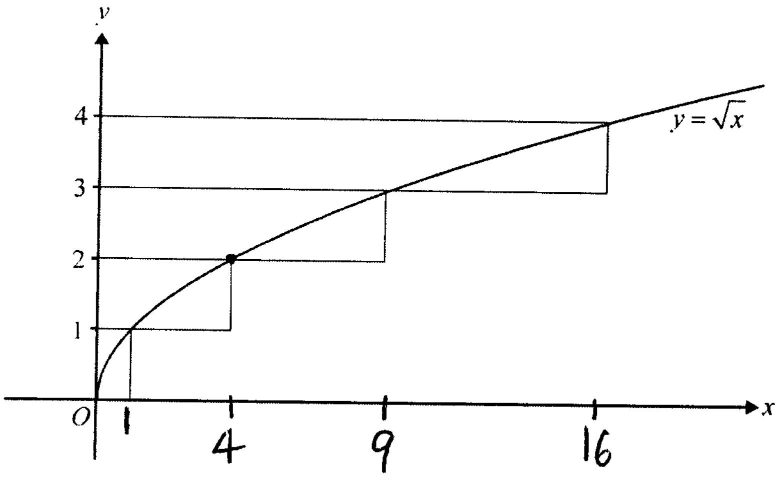
$= \dfrac{\frac{3}{7}}{\frac{29}{35}}$

$= \frac{3}{7} \times \frac{35}{29} = \frac{15}{29}$

Question 14

The graph of $f: R^+ \cup \{0\} \to R,\ f(x) = \sqrt{x}$ is shown below.

In order to find an approximation to the area of the region bounded by the graph of f, the y-axis and the line $y = 4$, Zoe draws four rectangles, as shown, and calculates their total area.



Zoe's approximation to the area of the region is

A. 14

B. 21

C. 29

D. 30

E. $\frac{64}{3}$

$1 \times 1 + 1 \times 4 + 1 \times 9 + 1 \times 16$

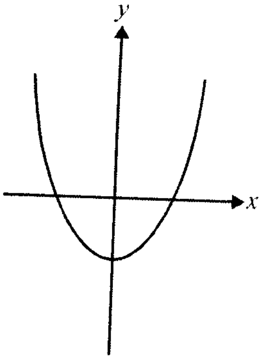
$= 1 + 4 + 9 + 16$

$= 30$

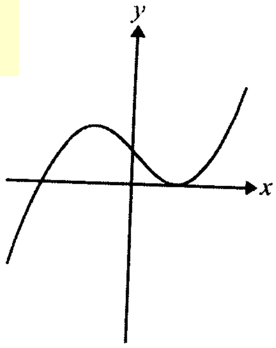
Question 15

If $f'(x) = 3x^2 - 4$, which one of the following graphs could represent the graph of $y = f(x)$?

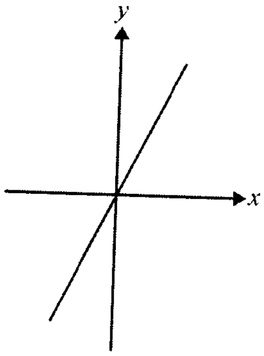
A.



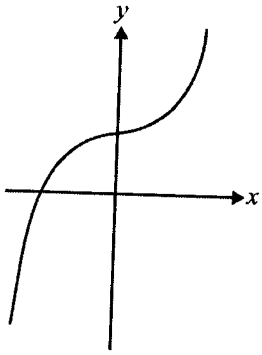
B.



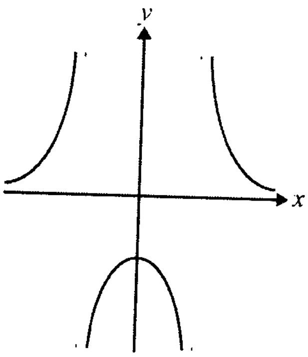
C.



D.



E.



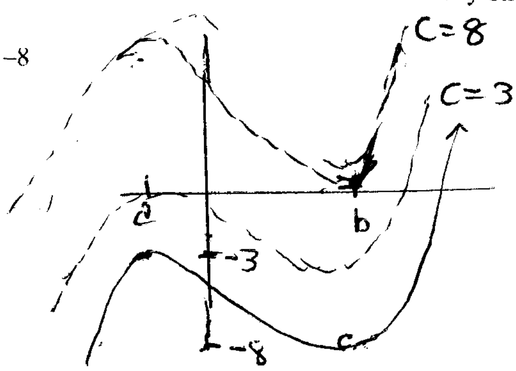
f must be a cubic with two stationary points symmetrically positioned with respect to the y-axis since $f'(x)=0$ gives $3x^2=4$

$x = \pm\sqrt{\frac{4}{3}}$

Question 16

The graph of a cubic function f has a local maximum at $(a, -3)$ and a local minimum at $(b, -8)$.
The values of c, such that the equation $f(x) + c = 0$ has exactly one solution, are

A. $3 < c < 8$
B. $c > -3$ or $c < -8$
C. $-8 < c < -3$
D. $c < 3$ or $c > 8$
E. $c < -8$



One x-intercept for

$c < 3 \cup c > 8$

Question 17

A system of simultaneous linear equations is represented by the matrix equation

$$\begin{bmatrix} m & 3 \\ 1 & m+2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ m \end{bmatrix}.$$

$m(m+2)-3=0$

$m^2+2m-3=0$

$(m+3)(m-1)=0$

$m=-3, 1$

The system of equations will have **no solution** when

A. $m = 1$

B. $m = -3$

C. $m \in \{1, -3\}$

D. $m \in R\backslash\{1\}$

E. $m \in \{1, 3\}$

If $m=1$: $\left.\begin{array}{l} x+3y=1 \\ x+3y=1 \end{array}\right]$ same eqn.

If $m=-3$: $\left.\begin{array}{l} -3x+3y=1 \\ x-y=-3 \end{array}\right]$ parallel lines

$\therefore m=-3$ gives no solution

Question 18

The tangent to the graph of $y = \log_e(x)$ at the point $(a, \log_e(a))$ crosses the x-axis at the point $(b, 0)$, where $b < 0$.

Which of the following is **false**?

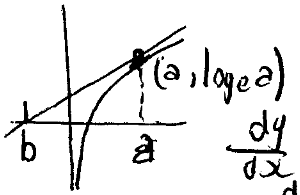
A. $1 < a < e$

B. The gradient of the tangent is positive

C. $a > e$

D. The gradient of the tangent is $\frac{1}{a}$

E. $a > 0$



$\frac{dy}{dx} = \frac{1}{x}$

$\left.\frac{dy}{dx}\right|_{x=a} = \frac{1}{a}$

$\therefore \frac{1}{a} = \frac{\log_e a - 0}{a-b}$

$\therefore a-b = a\log_e a$

$a - a\log_e a = b$

But $b<0$ $\therefore a - a\log_e a < 0$

$\therefore 1 < \log_e a$ $\therefore a > e$.

Question 19

A function f has the following two properties for all real values of θ.

$$f(\pi-\theta) = -f(\theta) \text{ and } f(\pi-\theta) = -f(-\theta)$$

A possible rule for f is

A. $f(x) = \sin(x)$

B. $f(x) = \cos(x)$

C. $f(x) = \tan(x)$

D. $f(x) = \sin\left(\frac{x}{2}\right)$

E. $f(x) = \tan(2x)$

$f(\pi-\theta) = -f(\theta)$, $f(\pi-\theta) = -f(-\theta)$

Test: $y=\sin x$ $\sin(\pi-\theta) = \sin\theta$ $\therefore$ Does not work

Test: $y=\cos x$ $\cos(\pi-\theta) = -\cos\theta$ True

$\cos(\pi-\theta) = -\cos(-\theta)$ True

since $\cos(-\theta) = \cos(\theta)$

Question 20

A discrete random variable X has the probability function $\Pr(X = k) = (1-p)^k p$, where k is a non-negative integer.

$\Pr(X > 1)$ is equal to

A. $1 - p + p^2$

B. $1 - p^2$

C. $p - p^2$

D. $2p - p^2$

E. $(1 - p)^2$

$\Pr(X>1) = 1 - \Pr(X=0) - \Pr(X=1)$

$= 1 - (1-p)^0 \cdot p - (1-p)^1 p$

$= 1 - p - p(1-p)$

$= 1 - p - p + p^2$

$= 1 - 2p + p^2$

$= (1-p)^2$

Question 21

Q21 involves related rates and is no longer on the course

Question 22

Q22 involves modulus and so is no longer on the course

END OF SECTION 1

TURN OVER

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

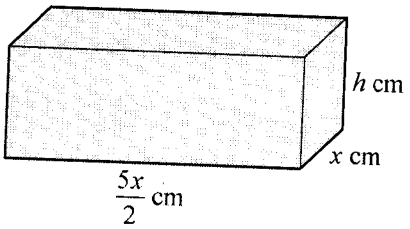
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

A solid block in the shape of a rectangular prism has a base of width x cm. The length of the base is two-and-a-half times the width of the base.



The block has a total surface area of 6480 sq cm.

a. Show that if the height of the block is h cm, $h = \dfrac{6480 - 5x^2}{7x}$.

Let S = Total Surface Area

$S = 2\left(\frac{5x}{2}\right)(x) + 2hx + 2\left(\frac{5x}{2}\right)h$

$S = 5x^2 + 2hx + 5xh$

$S = 5x^2 + 7xh$

But $S = 6480$

$6480 = 5x^2 + 7xh$

$\therefore \quad h = \dfrac{6480 - 5x^2}{7x}$

2 marks

b. The volume, V cm^3, of the block is given by $V(x) = \dfrac{5x(6480 - 5x^2)}{14}$.
Given that $V(x) > 0$ and $x > 0$, find the possible values of x.

For $V > 0$ we must have:

$$6480 - 5x^2 > 0$$

$$\therefore x^2 < \frac{6480}{5}$$

$$x^2 < 1296$$

$$\therefore x < \sqrt{1296} = 36$$

Domain: $x \in (0, 36)$

2 marks

c. Find $\dfrac{dV}{dx}$, expressing your answer in the form $\dfrac{dV}{dx} = ax^2 + b$, where a and b are real numbers.

$$V(x) = \frac{5}{14}(6480x - 5x^3)$$

$$V'(x) = \frac{5}{14}(6480 - \quad)$$

$$\therefore \frac{dV}{dx} = \frac{16200}{7} - \frac{75x^2}{14}$$

3 marks

d. Find the exact values of x and h if the block is to have maximum volume.

For a maximum, $\dfrac{dV}{dx} = 0$

$$\therefore \frac{16200}{7} - \frac{75x^2}{14} = 0$$

$$\therefore x = 12\sqrt{3} \quad \left(\text{reject } x = -12\sqrt{3} \text{ as } x > 0\right)$$

2 marks

Then, since $h = \dfrac{6480 - 5x^2}{7x}$,

When $x = 12\sqrt{3}$,

$$h = \frac{120\sqrt{3}}{7}$$

Question 2

Let $f: R\backslash\{2\} \rightarrow R$, $f(x) = \dfrac{1}{2x-4} + 3$.

a. Sketch the graph of $y = f(x)$ on the set of axes below. Label the axes intercepts with their coordinates and label each of the asymptotes with its equation.

y-int: $\left(0, \frac{11}{4}\right)$

x-int:

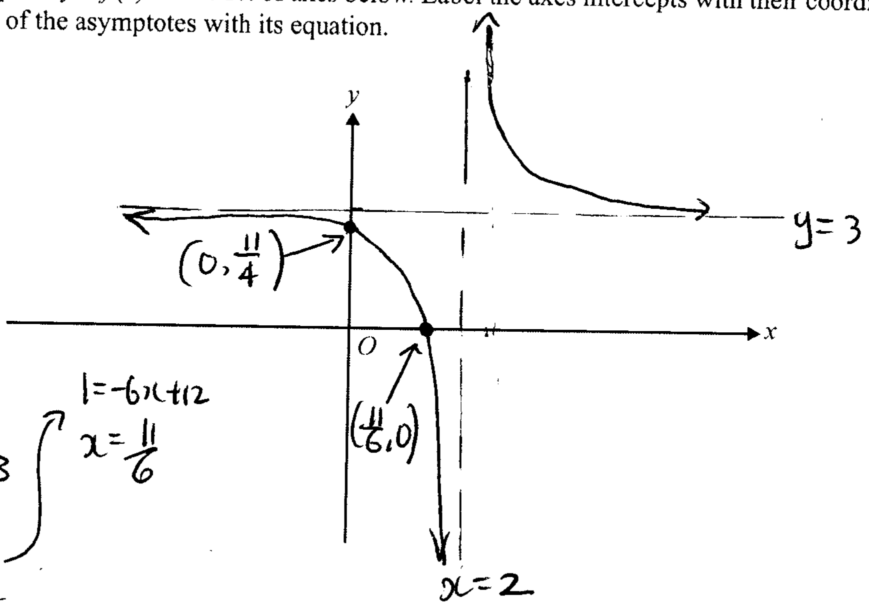
$\frac{1}{2x-4} + 3 = 0$

$-\frac{1}{2x-4} = 3$

$\frac{1}{2x-4} = -3$

$1 = -6x + 12$

$x = \frac{11}{6}$



3 marks

b. **i.** Find $f'(x)$.

$f'(x) = -\dfrac{1}{2(x-2)^2}$

ii. State the range of f'.

$(-\infty, 0)$

iii. Using the result of **part ii.** explain why f has no stationary points.

$f'(x)$ is never equal to zero.

1 + 1 + 1 = 3 marks

c. If (p, q) is any point on the graph of $y = f(x)$, show that the equation of the tangent to $y = f(x)$ at this point can be written as $(2p-4)^2(y-3) = -2x + 4p - 4$.

At (p, q), $f'(p) = -\frac{1}{2(p-2)^2}$

$q = \frac{1}{2p-4} + 3 \quad \therefore \quad m = -\frac{1}{2(p-2)^2}, \ (x_1, y_1) = \left(p, \frac{1}{2p-4} + 3\right)$

Equation of tangent:

$$y - \left(\frac{1}{2p-4} + 3\right) = -\frac{1}{2(p-2)^2}(x - p)$$

$$y - 3 = -\frac{1}{2(p-2)^2}(x-p) + \frac{1}{2p-4}$$

$$\therefore 2(p-2)^2(y-3) = -(x-p) + \frac{2(p-2)^2}{2(p-2)}$$

$$2(p-2)^2(y-3) = -(x-p) + p - 2$$

$$2(p-2)^2(y-3) = -x + 2p - 2$$

2 marks

$$\therefore 4(p-2)^2(y-3) = -2x + 4p - 4$$

$$\therefore (2p-4)^2(y-3) = -2x + 4p - 4$$

d. Find the coordinates of the points on the graph of $y = f(x)$ such that the tangents to the graph at these points intersect at $\left(-1, \frac{7}{2}\right)$.

If tangent goes through $\left(-1, \frac{7}{2}\right)$ then these co-ordinates satisfy the tangent line equation.

$$\therefore \quad (2p-4)^2\left(\frac{7}{2}-3\right) = -2\times -1 + 4p - 4$$

$$(2p-4)^2 \times \frac{1}{2} = 2 + 4p - 4$$

$$\frac{(2p-4)^2}{2} = 4p - 2$$

$$\frac{4(p-2)^2}{2} = 4p-2$$

$$2(p-2)^2 = 4p-2$$

$$(p-2)^2 = 2p-1$$

4 marks

$$p^2 - 4p + 4 = 2p - 1$$

$$p^2 - 6p + 5 = 0$$

$$(p-5)(p-1) = 0 \quad \therefore p = 1, 5$$

If $p=1$, then the point on the curve is: $\left(1, \frac{1}{2\times1-4}+3\right)$
$= \left(1, \frac{5}{2}\right)$

If $p = 5$, then the point on the curve is $\left(5, \frac{1}{2\times5-4}+3\right)$
$= \left(5, \frac{19}{6}\right)$

e. A transformation $T: R^2 \to R^2$ that maps the graph of f to the graph of the function $g: R\backslash\{0\} \to R, g(x) = \frac{1}{x}$ has rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$, where a, c and d are non-zero real numbers.

Find the values of a, c and d.

Image: $y' = \frac{1}{x'}$

Original $y = \frac{1}{2x-4} + 3$

$\therefore \quad y - 3 = \frac{1}{2x-4}$

$y' = y - 3$

$x' = 2x - 4$

$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \end{bmatrix}$

2 marks

$a = 2$

$c = -4$

$d = -3.$

Question 3

Steve, Katerina and Jess are three students who have agreed to take part in a psychology experiment. Each student is to answer several sets of multiple-choice questions. Each set has the same number of questions, n, where n is a number greater than 20. For each question there are four possible options (A, B, C or D), of which only one is correct.

a. Steve decides to guess the answer to every question, so that for each question he chooses A, B, C or D at random.

Let the random variable X be the number of questions that Steve answers correctly in a particular set.

i. What is the probability that Steve will answer the first three questions of this set correctly?

$\frac{1}{4}$ C $\frac{1}{4}$ C $\frac{1}{4}$ C

Required probability $= \left(\frac{1}{4}\right)^3 = \frac{1}{64}$

ii. Find, to four decimal places, the probability that Steve will answer at least 10 of the first 20 questions of this set correctly.

$X \overset{d}{=} Bi(n = 20, p = \frac{1}{4})$

$Pr(X \geq 10) = 0.0139$

iii. Use the fact that the variance of X is $\frac{75}{16}$ to show that the value of n is 25.

$npq = \frac{75}{16}$

$\therefore n \times \frac{1}{4} \times \frac{3}{4} = \frac{75}{16}$

$\frac{3n}{16} = \frac{75}{16}$

$3n = 75$

$n = 25$

1 + 2 + 1 = 4 marks

c. The probability that Jess will answer any question correctly, independently of her answer to any other question, is p ($p > 0$). Let the random variable Y be the number of questions that Jess answers correctly in any set of 25.

If $\Pr(Y > 23) = 6\Pr(Y = 25)$, show that the value of p is $\frac{5}{6}$.

$Y \overset{d}{=} Bi(n=25, p) \quad Y = 0, 1, 2, \ldots, 25$

$\Pr(Y>23) = \Pr(Y=24) + \Pr(Y=25)$

$\therefore \Pr(Y=24) + \Pr(Y=25) = 6\Pr(Y=25)$

$\therefore \Pr(Y=24) = 5\Pr(Y=25)$

$\therefore \binom{25}{24} p^{24}(1-p) = 5 \times \binom{25}{25} p^{25}(1-p)^0$

Full working must be shown since this is a "show that" question

$\therefore 25p^{24}(1-p) = 5p^{25}$

$\therefore 25(1-p) = 5p$

$5 - 5p = p$

$6p = 5 \quad \therefore p = \frac{5}{6}$

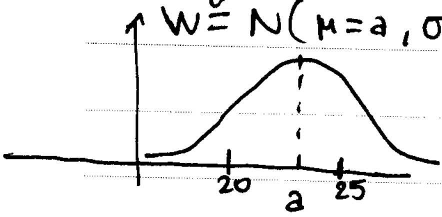
2 marks

d. From these sets of 25 questions being completed by many students, it has been found that the time, in minutes, that any student takes to answer each set of 25 questions is another random variable, W, which is **normally distributed** with mean a and standard deviation b.

It turns out that, for Jess, $\Pr(Y \geq 18) = \Pr(W \geq 20)$ and also $\Pr(Y \geq 22) = \Pr(W \geq 25)$.

Calculate the values of a and b, correct to three decimal places.

$W \overset{d}{=} N(\mu = a, \sigma = b)$



$\Pr(Y \geq 18) = 0.9552681$

$\Pr(Y \geq 22) = 0.38156649$

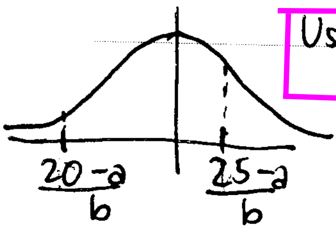
$\Pr(W \geq 20) = 0.9552681$

$\Pr(W \geq 25) = 0.38156649$

$Z = \frac{W-a}{b}$

$\Pr\left(Z \geq \frac{20-a}{b}\right) = 0.9552681 \quad \therefore \Pr\left(Z \leq \frac{20-a}{b}\right) = 0.04473193$

$\Pr\left(Z \geq \frac{25-a}{b}\right) = 0.38156649 \quad \Pr\left(Z \leq \frac{25-a}{b}\right) = 0.61843351$



Use invNorm

$\frac{20-a}{b} = \text{invNorm}(0, 1, 0.04473193)$

$\frac{25-a}{b} = \text{invNorm}(0, 1, 0.61843351)$

4 marks

CAS steps only...do NOT write as working

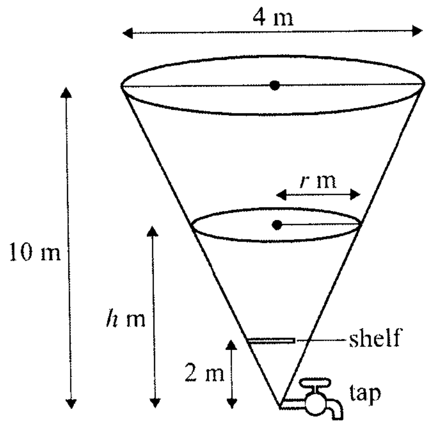
$\frac{20-a}{b} = -1.6982322$
$\frac{25-a}{b} = 0.3013692$ } Solve for a and b

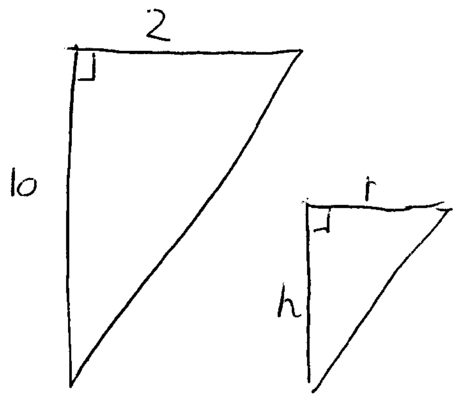
$a \simeq 24.246,$
$b \simeq 2.500$

Question 4

Tasmania Jones is in the jungle, searching for the Quetzalotl tribe's valuable emerald that has been stolen and hidden by a neighbouring tribe. Tasmania has heard that the emerald has been hidden in a tank shaped like an inverted cone, with a height of 10 metres and a diameter of 4 metres (as shown below).

The emerald is on a shelf. The tank has a poisonous liquid in it.





a. If the depth of the liquid in the tank is h metres

i. find the radius, r metres, of the surface of the liquid in terms of h

$$\frac{h}{10} = \frac{r}{2}$$

$$h = 5r$$

$$\therefore r = \frac{h}{5}$$

ii. show that the volume of the liquid in the tank is $\frac{\pi h^3}{75}$ m^3.

$$V = \frac{\pi r^2 h}{3}$$

But $r = \frac{h}{5}$

1 + 1 = 2 marks

$$\therefore V = \frac{\pi}{3} \times \left(\frac{h}{5}\right)^2 \times h$$

$$\therefore V = \frac{\pi h^3}{75}$$

The tank has a tap at its base that allows the liquid to run out of it. The tank is initially full. When the tap is turned on, the liquid flows out of the tank at such a rate that the depth, h metres, of the liquid in the tank is given by

$$h = 10 + \frac{1}{1600}(t^3 - 1200t),$$

where t minutes is the length of time after the tap is turned on until the tank is empty.

b. Show that the tank is empty when $t = 20$.

When $t = 20$,

$$h = 10 + \frac{1}{1600}(20^3 - 1200 \times 20)$$

$$= 10 + \frac{1}{1600}(8000 -$$

$$= 10 + \frac{-16000}{1600} = 0$$

1 mark

c. When $t = 5$ minutes, find

i. the depth of the liquid in the tank

$$h = 10 + \frac{1}{1600}(5^3 - 1200 \times 5)$$

$$= \frac{405}{64}$$

ii. the rate at which the volume of the liquid is decreasing, correct to one decimal place.

The rest of Q4 involves related rates and is no longer on the syllabus.

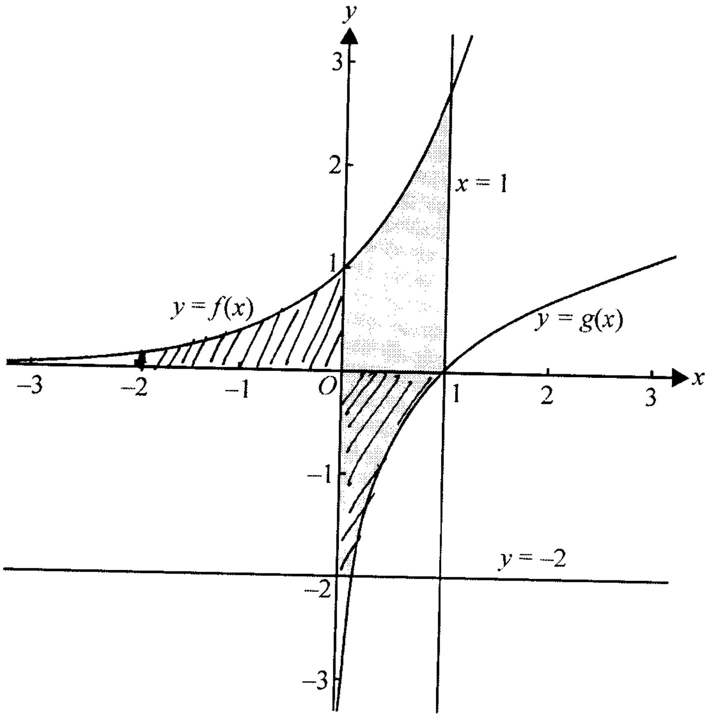
1 + 3 = 4 marks

Question 5

The shaded region in the diagram below is the plan of a mine site for the Black Possum mining company. All distances are in kilometres.

Two of the boundaries of the mine site are in the shape of the graphs of the functions

$$f: R \to R, f(x) = e^x \text{ and } g: R^+ \to R, g(x) = \log_e(x).$$



a. **i.** Evaluate $\int_{-2}^{0} f(x)dx$.

$$\int_{-2}^{0} e^{x} dx = \left[e^{x}\right]_{-2}^{0}$$

$$= e^{0} - e^{-2}$$

$$= 1 - \frac{1}{e^{2}}$$

ii. **Hence**, or otherwise, find the area of the region bounded by the graph of g, the x and y axes, and the line $y = -2$.

$$1 - \frac{1}{e^{2}}$$

(Areas are the same)

(since they are inverse functions they are exactly the same shape but rotated)

iii. Find the **total** area of the shaded region.

$$\int_0^1 e^x\,dx + 1 - \frac{1}{e^2}$$

$$= [e^x]_0^1 + 1 - \frac{1}{e^2}$$

$$= e - 1 + 1 - \frac{1}{e^2} = e - \frac{1}{e^2}$$

1 + 1 + 1 = 3 marks

b. The mining engineer, Victoria, decides that a better site for the mine is the region bounded by the graph of g and that of a new function $k: (-\infty, a) \to R$, $k(x) = -\log_e(a - x)$, where a is a positive real number.

i. Find, in terms of a, the x-coordinates of the points of intersection of the graphs of g and k.

$$-\log_e(a-x) = \log_e x$$

$$\therefore \log_e\left(\frac{1}{a-x}\right) = \log_e x, \quad 0 < x < a$$

$$\therefore \frac{1}{a-x} = x$$

$$ax - x^2 = 1$$

$$x^2 - ax + 1 = 0$$

$$x = \frac{a \pm \sqrt{a^2-4}}{2}$$

Both solutions are valid, since they lie in domain $(0, a)$.

Use Quadratic formula to solve!

ii. **Hence**, find the set of values of a, for which the graphs of g and k have two distinct points of intersection.

For 2 distinct intersection points, $\Delta > 0$

$$\therefore a^2 - 4 > 0$$

$$\therefore a > 2 \quad (\text{since } a > 0)$$

2 + 1 = 3 marks

c. For the new mine site, the graphs of g and k intersect at two distinct points, A and B. It is proposed to start mining operations along the line segment AB, which joins the two points of intersection.

Victoria decides that the graph of k will be such that the x-coordinate of the midpoint of AB is $\sqrt{2}$. Find the value of a in this case.

The x co-ordinate of A is $\frac{a-\sqrt{a^2-4}}{2}$

and for B is:

$\frac{a+\sqrt{a^2-4}}{2}$

$\therefore \frac{1}{2}\left(\frac{a-\sqrt{a^2-4}}{2}+\frac{a+\sqrt{a^2-4}}{2}\right)=\sqrt{2}$

$\therefore \frac{a}{2}=\sqrt{2}$

$a=2\sqrt{2}$

2 marks

END OF QUESTION AND ANSWER BOOK