QUESTIONS FROM 2017 VCAA EXAMS ON PROBABILITY

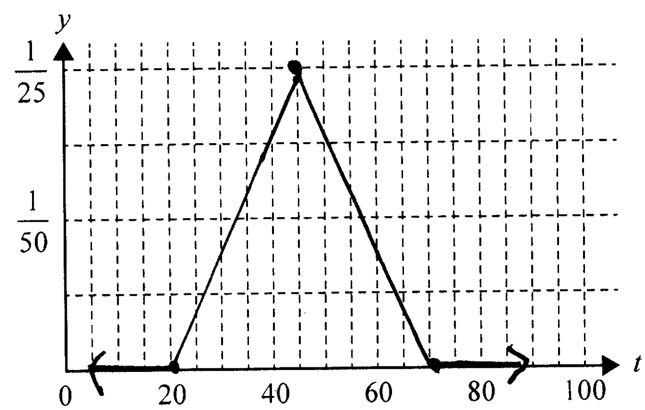
Question 3 (19 marks)

The time Jennifer spends on her homework each day varies, but she does some homework every day.

The continuous random variable T, which models the time, t, in minutes, that Jennifer spends each day on her homework, has a probability density function f, where

$$f(t)=\begin{cases}\frac{1}{625}(t-20) & 20\le t<45\\ \frac{1}{625}(70-t) & 45\le t\le 70\\ 0 & \text{elsewhere}\end{cases}$$

a. Sketch the graph of f on the axes provided below. 3 marks



b. Find $\Pr(25 \le T \le 55)$. 2 marks

$$\Pr(25\le T\le 55)=\int_{25}^{55} f(t)\,dt$$

$$=\frac{4}{5}$$

c. Find $\Pr(T \le 25 \mid T \le 55)$. 2 marks

$$\Pr(T\le 25 \mid T\le 55)=\frac{\Pr(T\le 25)}{\Pr(T\le 55)}$$

$$=\frac{\frac{1}{50}}{0.82}=\frac{1}{50}\times\frac{50}{41}$$

$$=\frac{1}{41}$$

d. Find a such that $\Pr(T \geq a) = 0.7$, correct to four decimal places. 2 marks

$$\int_a^{70} f(t)\,dt = 0.7$$

$$a = 39.3649$$

e. The probability that Jennifer spends more than 50 minutes on her homework on any given day is $\frac{8}{25}$. Assume that the amount of time spent on her homework on any day is independent of the time spent on her homework on any other day.

i. Find the probability that Jennifer spends more than 50 minutes on her homework on more than three of seven randomly chosen days, correct to four decimal places. 2 marks

X = no. of days she spends more than 50 min on homework

$X \sim Bi\left(n=7,\ p=\frac{8}{25}\right)$ $\Pr(X>3) = \Pr(X \geq 4) \simeq 0.1534$

ii. Find the probability that Jennifer spends more than 50 minutes on her homework on at least two of seven randomly chosen days, given that she spends more than 50 minutes on her homework on at least one of those days, correct to four decimal places. 2 marks

$$\Pr(X \geq 2 \mid X \geq 1)$$

$$= \frac{\Pr(X \geq 2 \cap X \geq 1)}{\Pr(X \geq 1)}$$

$$= \frac{\Pr(X \geq 2)}{\Pr(X \geq 1)}$$

$$= 0.7626$$

Let p be the probability that on any given day Jennifer spends more than d minutes on her homework.

Let q be the probability that on two or three days out of seven randomly chosen days she spends more than d minutes on her homework.

f. Express q as a polynomial in terms of p. 2 marks

Let Y = no. of days she spends more than d minutes on homework $Y \sim Bi(n=7,\ p)$

$q = Pr(Y=2) + Pr(Y=3)$

$q = \binom{7}{2} p^2 (1-p)^5 + \binom{7}{3} p^3 (1-p)^4 = 21p^2(1-p)^5 + 35p^3(1-p)^4$

g. **i.** Find the maximum value of q, correct to four decimal places, and the value of p for which this maximum occurs, correct to four decimal places. 2 marks

Let $q = g(p)$ $\quad g(p) = 21p^2(1-p)^5 + 35p^3(1-p)^4$

For a maximum, $g'(p) = 0 \therefore p \approx 0.3539$, $q_{max} = g(0.3539) \approx 0.5665$

ii. Find the value of d for which the maximum found in **part g.i.** occurs, correct to the nearest minute. 2 marks

$Pr(T \geq d) = 0.35388...$

$\therefore \int_d^{70} f(t)\,dt = 0.35388...$

Solving: $d \approx 48.9676..$

$\therefore d \simeq 49$ minutes

Question 5

The 95% confidence interval for the proportion of ferry tickets that are cancelled on the intended departure day is calculated from a large sample to be (0.039, 0.121).

The sample proportion from which this interval was constructed is

A. 0.080
B. 0.041
C. 0.100
D. 0.062
E. 0.059

$\hat{p}$ is at midpoint of interval

$\hat{p} = \dfrac{0.039 + 0.121}{2} = 0.08$

Question 16

For random samples of five Australians, $\hat{P}$ is the random variable that represents the proportion who live in a capital city.

Given that $\Pr(\hat{P} = 0) = \frac{1}{243}$, then $\Pr(\hat{P} > 0.6)$, correct to four decimal places, is

A. 0.0453
B. 0.3209
C. 0.4609
D. 0.5390
E. 0.7901

$\Pr(\hat{P} > 0.6)$
$= \Pr(X > 3)$
$= \Pr(X=4) + \Pr(X=5) \approx 0.4609$
[Use binomcdf$(5, \frac{2}{3}, 4, 5)$]

$\hat{P} = \frac{X}{5}$
$X \sim Bi(n=5, p=?)$
$\Pr(\hat{P}=0) = \Pr(X=0)$
$= \binom{5}{0} p^0 (1-p)^5 = \frac{1}{243}$
$\therefore 1-p = \frac{1}{3}$
$\therefore p = \frac{2}{3}$

Question 18

Let X be a discrete random variable with binomial distribution $X \sim Bi(n, p)$. The mean and the standard deviation of this distribution are equal.

Given that $0 < p < 1$, the smallest number of trials, n, such that $p \le 0.01$ is

A. 37
B. 49
C. 98
D. 99
E. 101

$np = \sqrt{np(1-p)}$
$n^2p^2 = np(1-p)$
$\therefore np = 1-p$
$np + p = 1$
$\therefore p = \frac{1}{1+n}$

$\frac{1}{1+n} \le \frac{1}{100}$
$\therefore n+1 \ge 100$
$n \ge 99$

Question 14

The random variable X has the following probability distribution, where $0 < p < \frac{1}{3}$.

x	−1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

The variance of X is

A. $2p(1-3p)$
B. $1-4p$
C. $(1-3p)^2$
D. $6p-16p^2$
E. $p(5-9p)$

$Var(X) = E(X^2) - \mu^2$
$\mu = -1 \times p + 0 \times 2p + 1 \times (1-3p)$
$= 1 - 3p - p = 1 - 4p$
$E(X^2) = (-1)^2 \times p + 0^2 \times 2p + 1^2 \times (1-3p)$
$= 1 - 2p$
$\therefore Var(X) = 1 - 2p - (1-4p)^2 = 1 - 2p - (1 - 8p + 16p^2)$
$= 6p - 16p^2$

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Question 8 (5 marks)

For events A and B from a sample space, $\Pr(A \mid B) = \frac{1}{5}$ and $\Pr(B \mid A) = \frac{1}{4}$. Let $\Pr(A \cap B) = p$.

a. Find $\Pr(A)$ in terms of p. 1 mark

$$\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

$$\therefore \frac{1}{4} = \frac{p}{\Pr(A)} \quad (\Pr(A) \neq 0)$$

$$\Pr(A) = 4p$$

b. Find $\Pr(A' \cap B')$ in terms of p. 2 marks

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad \therefore \frac{1}{5} = \frac{p}{\Pr(B)}$$

$$\therefore \Pr(B) = 5p$$

	B	B'	
A	p	$3p$	$4p$
A'	$4p$	$1-8p$	$1-4p$
	$5p$	$1-5p$	1

$$\Pr(A' \cap B') = 1 - 8p$$

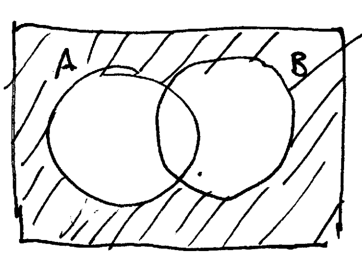
c. Given that $\Pr(A \cup B) \leq \frac{1}{5}$, state the largest possible interval for p. 2 marks

$$\therefore 1 - 8p \geq \frac{4}{5}$$

$$\frac{1}{5} \geq 8p$$

$$p \leq \frac{1}{40}$$

But $p > 0$ $\quad \therefore \quad 0 < p \leq \frac{1}{40}$



$$\Pr(A' \cap B') = 1 - \Pr(A \cup B)$$

If $\Pr(A \cup B) \leq \frac{1}{5}$

then $\Pr(A' \cap B') \geq \frac{4}{5}$

PROBABILITY QUESTIONS FROM 2016 VCAA EXAMS

Question 3 (16 marks)

A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.

On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does **not** correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.

a. Determine the probability that at least one of the laptops is **not** correctly plugged into the trolley at the end of the lesson. Give your answer correct to four decimal places. 2 marks

Let X = no. of incorrectly plugged laptops

$X \sim Bi(n=22, p=0.1)$

$$\Pr(X \geq 1) = 1 - \Pr(X=0)$$

$$= 0.9015$$

b. A teacher observes that at least one of the returned laptops is not correctly plugged into the trolley.

Given this, find the probability that fewer than five laptops are **not** correctly plugged in. Give your answer correct to four decimal places. 2 marks

$$\Pr(X<5 \mid X \geq 1)$$

$$= \Pr(X \leq 4 \mid X \geq 1)$$

$$= \frac{\Pr(1 \leq X \leq 4)}{\Pr(X \geq 1)}$$

$$= 0.9311$$

The time for which a laptop will work without recharging (the battery life) is normally distributed, with a mean of three hours and 10 minutes and standard deviation of six minutes. Suppose that the laptops remain out of the recharging trolley for three hours.

c. For any one laptop, find the probability that it will stop working by the end of these three hours. Give your answer correct to four decimal places.

2 marks

$T \sim N\left(\mu = \frac{19}{6}, \sigma = 0.1\right)$

$6 \text{ min} = \frac{6}{60}$ hours

$10\text{min} = \frac{1}{6}$ hour

$\Pr(T \leq 3) = 0.0478$

A supplier of laptops decides to take a sample of 100 new laptops from a number of different schools. For samples of size 100 from the population of laptops with a mean battery life of three hours and 10 minutes and standard deviation of six minutes, $\hat{P}$ is the random variable of the distribution of sample proportions of laptops with a battery life of less than three hours.

d. Find the probability that $\Pr(\hat{P} \geq 0.06 \mid \hat{P} \geq 0.05)$. Give your answer correct to three decimal places. Do not use a normal approximation.

3 marks

Let X = no. of laptops with a battery life less than three hours

$X \sim Bi(n = 100, p = 0.0478)$

$\hat{p} = \frac{X}{100}$ $\quad \Pr(\hat{p} \geq 0.06 \mid \hat{p} \geq 0.05) = \frac{\Pr(\hat{p} \geq 0.06 \cap \hat{p} \geq 0.05)}{\Pr(\hat{p} \geq 0.05)}$

$= \frac{\Pr(\hat{p} \geq 0.06)}{\Pr(\hat{p} \geq 0.05)} = \frac{\Pr(X \geq 6)}{\Pr(X \geq 5)}$

$\simeq 0.658$

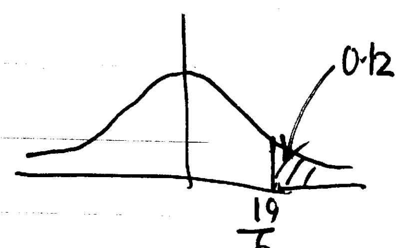
It is known that when laptops have been used regularly in a school for six months, their battery life is still normally distributed but the mean battery life drops to three hours. It is also known that only 12% of such laptops work for more than three hours and 10 minutes.

e. Find the standard deviation for the normal distribution that applies to the battery life of laptops that have been used regularly in a school for six months, correct to four decimal places.

2 marks

$T \sim N(\mu = 3, \sigma = ?)$

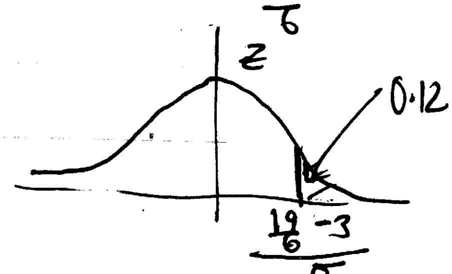
$\Pr\left(T > \frac{19}{6}\right) = 0.12$



$\therefore \Pr\left(Z > \frac{1}{6\sigma}\right) = 0.12$

$\therefore \frac{1}{6\sigma} \simeq 1.1749868$

$\sigma \simeq \frac{1}{6 \times 1.1749868} \simeq 0.1418$ hrs



(or 8.5107 minutes)

The laptop supplier collects a sample of 100 laptops that have been used for six months from a number of different schools and tests their battery life. The laptop supplier wishes to estimate the proportion of such laptops with a battery life of less than three hours.

f. Suppose the supplier tests the battery life of the laptops one at a time.

Find the probability that the first laptop found to have a battery life of less than three hours is the third one. 1 mark

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

The laptop supplier finds that, in a particular sample of 100 laptops, six of them have a battery life of less than three hours.

g. Determine the 95% confidence interval for the supplier's estimate of the proportion of interest. Give values correct to two decimal places. 1 mark

$\hat{p} = 0.06, n = 100$ (Use CAS 1-prop confidence interval)

$(0.01, 0.11)$

h. The supplier also provides laptops to businesses. The probability density function for battery life, x (in minutes), of a laptop after six months of use in a business is

$$f(x)=\begin{cases}\dfrac{(210-x)e^{\frac{x-210}{20}}}{400} & 0\le x\le 210\\ 0 & \text{elsewhere}\end{cases}$$

i. Find the **mean** battery life, in minutes, of a laptop with six months of business use, correct to two decimal places. 1 mark

$$E(X) = \int_0^{210} x f(x)\,dx$$

$$= \int_0^{210} \frac{x(210-x)}{400} e^{\frac{x-210}{20}}\,dx$$

$$\simeq 170.01$$

ii. Find the **median** battery life, in minutes, of a laptop with six months of business use, correct to two decimal places. 2 marks

$$\int_0^m \frac{(210-x)}{400} e^{\frac{x-210}{20}}\,dx = 0.5$$

Solving: $m \simeq 176.45 \quad (0<m<210)$

Question 7

The number of pets, X, owned by each student in a large school is a random variable with the following discrete probability distribution.

x	0	1	2	3
$\Pr(X = x)$	0.5	0.25	0.2	0.05

If two students are selected at random, the probability that they own the same number of pets is

A. 0.3
B. 0.305
C. 0.355
D. 0.405
E. 0.8

$(0.5)^2 + (0.25)^2 + (0.2)^2 + (0.05)^2$
$= 0.355$

Question 15

A box contains six red marbles and four blue marbles. Two marbles are drawn from the box, without replacement.

The probability that they are the same colour is

A. $\frac{1}{2}$

B. $\frac{28}{45}$

C. $\frac{7}{15}$

D. $\frac{3}{5}$

E. $\frac{1}{3}$

$\Pr(RR) + \Pr(BB)$
$= \frac{6}{10} \times \frac{5}{9} + \frac{4}{10} \times \frac{3}{9}$
$= \frac{30}{90} + \frac{12}{90} = \frac{42}{90}$
$= \frac{21}{45} = \frac{7}{15}$

Question 16

The random variable, X, has a normal distribution with mean 12 and standard deviation 0.25

If the random variable, Z, has the standard normal distribution, then the probability that X is greater than 12.5 is equal to

A. $\Pr(Z < -4)$
B. $\Pr(Z < -1.5)$
C. $\Pr(Z < 1)$
D. $\Pr(Z \geq 1.5)$
E. $\Pr(Z > 2)$

$Z = \frac{X - 12}{0.25} = 4(X-12)$

$\Pr(X > 12.5)$
$= \Pr(Z > 2)$

Question 17

Inside a container there are one million coloured building blocks. It is known that 20% of the blocks are red. A sample of 16 blocks is taken from the container. For samples of 16 blocks, $\hat{P}$ is the random variable of the distribution of sample proportions of red blocks. (Do not use a normal approximation.)

$\Pr\left(\hat{P} \geq \frac{3}{16}\right)$ is closest to

$p = 0.2 \qquad \hat{p} = \frac{x}{16}$

A. 0.6482

B. 0.8593

C. 0.7543

D. 0.6542

E. 0.3211

$\Pr\left(\hat{p} \geq \frac{3}{16}\right) = \Pr(X \geq 3)$

$X \sim Bi(n = 16, p = 0.2)$

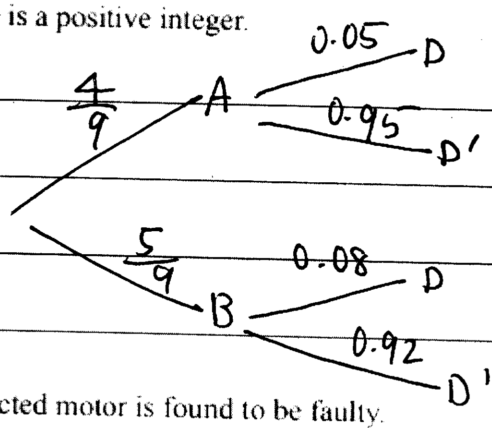
$\therefore \Pr(X \geq 3) = 0.6482$

CAS FREE QUESTIONS

Question 7 (3 marks)

A company produces motors for refrigerators. There are two assembly lines, Line A and Line B. 5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In one hour, 40 motors are produced from Line A and 50 motors are produced from Line B. At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

a. What is the probability that the selected motor is faulty? Express your answer in the form $\frac{1}{b}$, where b is a positive integer. 2 marks



$\Pr(D) = \frac{4}{9} \times \frac{5}{100} + \frac{5}{9} \times \frac{8}{100}$

$= \frac{1}{9} \times \frac{5}{25} + \frac{1}{9} \times \frac{8}{20}$

$= \frac{1}{9 \times 5} + \frac{1}{9} \times \frac{2}{5} = \frac{3}{45} = \frac{1}{15}$

b. The selected motor is found to be faulty.

What is the probability that it was assembled on Line A? Express your answer in the form $\frac{1}{c}$, where c is a positive integer. 1 mark

$\Pr(A|D) = \frac{\Pr(A \cap D)}{\Pr(D)} = \frac{\frac{4}{9} \times \frac{5}{100}}{\frac{1}{15}}$

$= \frac{\frac{1}{45}}{\frac{1}{15}} = \frac{1}{3}$